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Одеський національний університет імені І. І. Мечникова

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THE DYNAMICAL PROBLEM ON ACTING CONCENTRATED LOAD ON THE ELASTIC QUARTER SPACE

The wave field of an elastic quarter space is constructed when one face is rigidly fixed and a dynamic normal compressive load acts on the other along a rectangular section at the initial moment of time. Integral Laplace and Fourier transforms are applied sequentially to the equations of motion and boundary conditions in contrast to traditional approaches when integral transforms are applied to solutions' representations through harmonic functions. This leads to a one-dimensional vector homogeneous boundary value problem with respect to unknown displacement's transformants. The problem was solved using matrix differential calculus. The original displacement field was found after applying inverse integral transforms. For the case of stationary vibrations a method of calculating integrals in the solution in the near loading zone was indicated. For the analysis of oscillations in a remote zone the asymptotic formulas were constructed. The amplitude of vertical vibrations was investigated depending on the shape of the load section, natural frequencies of vibrations and the material of the medium.
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1. Introduction

During the construction and analysis of structures when dynamic or static loads appear, stress arise and concentrate in elastic bodies. These stresses can deform and even break the structure. Therefore they must be taken into account during design calculation. Because of this, problems of the elasticity theory appear in mathematical physics.

These problems were considered in a static and dynamic statements by many authors for different objects under different initial and boundary conditions [1–4]. An object such as a quarter space can be considered as a model before solving a similar problems for an infinite or semi-infinite layer and then for a plate. A quarter space is a special case of a spatial wedge. In particular for the second boundary value problem for a spatial wedge the exact solution

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was constructed by Ya. S. Uflyand [5]. In another work [6], the exact solution for the case, when normal displacements and tangential stress are given, was constructed. The exact solution of the mixed problem of the elasticity theory for a quarter-space in the static statement was found by G. Ya. Popov in [1]. It is essential that a new method was used in the solving of this problem, based on representation of new functions which are the sum of displacements' derivatives [7]. This method was successfully applied to solving Lamb problem [8]. Also using this method, homogeneous and inhomogeneous problems of the elasticity theory for a semifinite layer were solved [7]. The development of methods for problems of the elasticity theory for various objects, in particular for a quarter space, was also carried out by A. M. Alexandrov in [9]. A general solution for an elastic quarter space contact problem was presented in [10]. Dynamical stresses in elastic half-space were analysed in [11]. Plane contact problem on the pressure of a stamp with a rectangular base on a rough elastic halfspace was considered in [12].

Based on the results of [1; 8], as well as the method of representing the equations of motion in terms of two jointly and one independently solvable equations, proposed in [7], the aim of this work is to obtain the exact formulas for displacements that appear in a quarter space when a dynamic compressive load acts on one of its faces.

2. Main Results

2.1. Statement of the problem.

An elastic quarter space $x > 0,\ -\infty < y < \infty,\ 0 < z < \infty,$ is considered. At the moment of time $t = 0$ dynamic normal load

$$\sigma_z\ (x, y, z, t)|_{z=0} = -p(x, y)P(t)$$

is applied to the boundary of the quarter space $z = 0$ across the rectangular area $0 \le x \le A,\ -B \le y \le B$ the tangential stresses over the entire XOY plane are zero. The face $x = 0$ is rigidly fixed. The nonstationary points' displacements of the quarter space $u(x, y, z, t),\ v(x, y, z, t),\ w(x, y, z, t)$ are required to be determined with zero initial conditions. The statement leads to the following boundary conditions

$$\sigma_z(x, y, 0, t) = -p(x, y)P(t),\ 0 \le x \le A;\ -B \le y \le B \qquad (1)$$

$$\sigma_z(x,y,0,t)=0,\ x>A;\ |y|>B$$
$$\tau_{zx}(x,y,0,t)=0,\ \tau_{zy}(x,y,0,t)=0$$
$$u(0,y,z,t)=v(0,y,z,t)=w(0,y,z,t)=0$$

The equations of motion in vector form are [8]

$$\Delta(u,v,w)+\frac{2}{\kappa-1}\left(\frac{\partial\Theta}{\partial x},\frac{\partial\Theta}{\partial y},\frac{\partial\Theta}{\partial z}\right)=\frac{\rho}{G}\left(\frac{\partial^2 u}{\partial t^2},\frac{\partial^2 v}{\partial t^2},\frac{\partial^2 w}{\partial t^2}\right), \tag{2}$$

where Δ is the Laplace operator, $\kappa = 3-4\mu$, μ — Poisson's ratio, $\Theta = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$ — volume expansion, ρ — density of the medium material, G — shear modulus; $\frac{\rho}{G}=\frac{1}{c^2}$, c — wave propagation speed.

To obtain a solution to the posed problem, it is enough firstly to obtain a solution when the dynamic force concentrated at an arbitrary point (a,b) of the face $z=0$, and then distribute it over the required section, i.e.

$$p(x,y)=\delta(x-a)\delta(y-b)$$

Let's set up a dimensionless coordinate system

$$\tilde{x}=\frac{x}{a},\ \tilde{y}=\frac{(y-b)}{a},\ \tilde{z}=\frac{z}{a},\ \tilde{t}=\left(\frac{1}{c^2}\right)t \tag{3}$$

Further, the “waves” are omitted, implying the replacement (3), introduce the new functions [7]

$$\begin{aligned} Z(x,y,z)&=\frac{\partial}{\partial x}u(x,y,z)+\frac{\partial}{\partial y}v(x,y,z)\\ \widetilde{Z}(x,y,z)&=\frac{\partial}{\partial x}v(x,y,z)-\frac{\partial}{\partial y}u(x,y,z) \end{aligned} \tag{4}$$

Then the system of equations of motion (2) and the boundary conditions (1) are rewritten in the form relatively new functions.

$$\begin{cases} \Delta W+\dfrac{2}{\kappa-1}\dfrac{\partial}{\partial z}\left(Z+\dfrac{\partial W}{\partial z}\right)=\dfrac{\partial^2 W}{\partial t^2}\\ \Delta Z+\dfrac{2}{\kappa-1}\nabla_{xy}\left(Z+\dfrac{\partial W}{\partial z}\right)=\dfrac{\partial^2 Z}{\partial t^2} \end{cases} \tag{5}$$

$$\Delta\widetilde{Z}=\frac{\partial^2\widetilde{Z}}{\partial t^2} \tag{6}$$

$$\begin{aligned}
\mu Z(x,y,0,t) + (1-\mu)\frac{\partial}{\partial z}W(x,y,0,t) &= -\frac{\kappa-1}{4Ga}\delta(x-1)\delta(y)P(t)\\
\nabla_{xy}W(x,y,0,t) + \frac{\partial}{\partial z}Z(x,y,0,t) &= 0\\
\frac{\partial}{\partial z}\tilde{Z}(x,y,0,t) &= 0\\
u(0,y,z,t) = v(0,y,z,t) &= w(0,y,z,t) = 0
\end{aligned} \tag{7}$$

where $\nabla_{xy} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

The original initial boundary value problem takes the form (5)-(7) under the initial conditions

$$\left[W, Z, \widetilde{Z}\right]\Big|_{t=0} = 0 \quad \frac{\partial}{\partial t}\left[W, Z, \widetilde{Z}\right]\Big|_{t=0} = 0 \tag{8}$$

After finding the unknown functions $W, Z, \widetilde{Z}$ the Poisson equations should be solved in order to determine the displacements u and v

$$\nabla_{xy}u = \frac{\partial}{\partial x}Z - \frac{\partial}{\partial y}\widetilde{Z},\ \nabla_{xy}v = \frac{\partial}{\partial y}Z + \frac{\partial}{\partial x}\widetilde{Z} \tag{9}$$

2.2. Reducing the problem to a vector one-dimensional problem

The Fourier transform with respect to the variable y, sin - transform with respect to the variable x and the Laplace transform with respect to the variable t, with parameters β, α and p respectively are applied successively to (5), (6).

$$\begin{bmatrix} W_{\alpha\beta p}(z) \\ Z_{\alpha\beta p}(z) \end{bmatrix} = \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} \begin{bmatrix} W(x,y,z,t) \\ Z(x,y,z,t) \end{bmatrix} e^{i\beta y}\sin\alpha x\, e^{-pt}\, dy\, dx\, dt \tag{10}$$

The following conditions are assumed to be additionally satisfied [1]

$$Z_\beta(0,z) = 0,\ \widetilde{Z}_\beta(0,z) = 0 \tag{11}$$

The function $\widetilde{Z}_{\alpha\beta p}(z)$ satisfies the homogeneous problem

$$\widetilde{Z}''_{\alpha\beta p}(z) - (N^2+p^2)\widetilde{Z}_{\alpha\beta p}(z) = 0,\ 0 < z < \infty,\ \widetilde{Z}'_{\alpha\beta p}(0) = 0 \tag{12}$$

and therefore $\widetilde{Z}(x,y,z,t) \equiv 0$. The system of equations (5) and the boundary conditions (7) take the form

$$\begin{cases} W''_{\alpha\beta p}(z) + \dfrac{2}{\kappa+1}Z'_{\alpha\beta p}(z) - N^2\dfrac{\kappa-1}{\kappa+1}W_{\alpha\beta p}(z) - \dfrac{\kappa-1}{\kappa+1}p^2 W_{\alpha\beta p} = 0 \\ Z''_{\alpha\beta p}(z) - \dfrac{2}{\kappa-1}N^2 W'_{\alpha\beta p}(z) - N^2\dfrac{\kappa+1}{\kappa-1}Z_{\alpha\beta p}(z) - p^2 Z_{\alpha\beta p}(z) = 0 \end{cases} \tag{13}$$

$$
\begin{gathered}
-N^2 W_{\alpha\beta p}(0) + Z'_{\alpha\beta p} = 0 \\
(3-\kappa)\mu Z_{\alpha\beta p}(0) + (1-\mu) W'_{\alpha\beta p}(0) = -\frac{\kappa-1}{4Ga}\cdot \sin\alpha \cdot P_p \\
P_p = \int\limits_0^\infty P(t) e^{-pt} dt;\ N^2 = \alpha^2 + \beta^2;
\end{gathered}
\tag{14}
$$

To rewrite the system (13) in vector form, the unknown vector of the displacement's transformant is introduced

$$
\vec{\mathbf{y}}(z) = \begin{pmatrix} W_{\alpha\beta p}(z) \\ Z_{\alpha\beta p}(z) \end{pmatrix}
$$

as well matrices

$$
\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \mathbf{Q} = \begin{pmatrix} 0 & \frac{1}{\kappa+1} \\ \frac{-N^2}{\kappa-1} & 0 \end{pmatrix},\ \mathbf{P} = \begin{pmatrix} \frac{\kappa-1}{\kappa+1} & 0 \\ 0 & \frac{\kappa-1}{\kappa+1} \end{pmatrix},\ \mathbf{T} = \begin{pmatrix} \frac{\kappa-1}{\kappa+1} & 0 \\ 0 & 1 \end{pmatrix}
$$

So, the system (13) takes form

$$
\mathrm{L}_2 \vec{\mathbf{y}}(z) = 0,\ 0 < z < \infty \tag{15}
$$

where the differential operator L_2 has a form

$$
\mathrm{L}_2 \vec{\mathbf{y}}(z) = \mathbf{I}\vec{\mathbf{y}}''(z) + 2\mathbf{Q}\vec{\mathbf{y}}'(z) - N^2 \mathbf{P}\vec{\mathbf{y}}(z) - p^2 \mathbf{T}\vec{\mathbf{y}}(z)
$$

The solution of the vector equation (15) is constructed on the basis of the matrix equation's solution $\mathrm{L}_2[\mathbf{Y}(z)] = 0$. The substitution $\mathbf{Y}(z) = e^{Nz}\mathbf{I}$ is made to form the characteristic matrix $\mathbf{M}(s) = \mathbf{I}s^2 + 2\mathbf{Q}s - N^2\mathbf{P} - p^2\mathbf{T}$. Inverse matrix has a form

$$
\mathbf{M^{-1}}(\mathbf{s}) = \frac{1}{\prod_{i=1}^4 (s - s_i)} \begin{pmatrix} s^2 - \frac{\kappa+1}{\kappa-1}N^2 - p^2 & -\frac{2s}{\kappa+1} \\ \frac{2s}{\kappa-1}N^2 & s^2 - N^2\frac{\kappa-1}{\kappa+1} - p^2\frac{\kappa-1}{\kappa+1} \end{pmatrix}
$$

$$
\begin{aligned}
s_1 &= -\sqrt{N^2 + \frac{\kappa-1}{\kappa+1}p^2}, & s_2 &= -\sqrt{N^2 + p^2}, \\
s_3 &= \sqrt{N^2 + \frac{\kappa-1}{\kappa+1}p^2}, & s_4 &= \sqrt{N^2 + p^2}.
\end{aligned}
$$

Here s_i $(i = \overline{1,4})$ are roots of the characteristic equation $\det[\mathbf{M}(s)] = 0$. The solution of the matrix equation is constructed by the formula [13]

$$
\mathbf{Y}_-(z) = \frac{1}{2\pi i} \oint\limits_C e^{sz} \mathbf{M}^{-1}(s) ds
$$

where C is a closed contour encompassing all zeros of the matrix's determinant $\mathbf{M}(s)$. The residues at the poles s_3 and s_4 give a growing solution at infinity and are therefore discarded. The residues at the poles s_1 and s_2 give a solution decreasing at infinity. After calculation a decreasing solution takes a form

$$\mathbf{Y}_-(z) = \frac{1}{2p^2}e^{-\Delta_1 z}\begin{pmatrix} \frac{(\kappa+1)N^2}{(\kappa-1)\Delta_1} & -1 \\ \frac{(\kappa+1)N^2}{(\kappa-1)} & -\Delta_1 \end{pmatrix} + \frac{1}{2p^2}e^{-\Delta_2 z}\begin{pmatrix} -\frac{(\kappa+1)}{(\kappa-1)}\Delta_2 & 1 \\ -\frac{(\kappa+1)}{(\kappa-1)}N^2 & \frac{N^2}{\Delta_2} \end{pmatrix} \tag{16}$$

where $\Delta_1 = \sqrt{N^2 + p^2},\ \Delta_2 = \sqrt{N^2 + \frac{p^2(\kappa-1)}{(\kappa+1)}}$

The solution of the vector equation (15) is constructed in the form

$$\vec{\mathbf{y}}(z) = \mathbf{Y}_-(z)\cdot\begin{pmatrix} C_0 \\ C_1 \end{pmatrix}$$

where constants $C_i,\ i = 0, 1$ are found by satisfying the boundary conditions (14). Thus, a system of linear algebraic equations is obtained

$$\begin{cases} \frac{\kappa+1}{\kappa-1}\frac{2N^2}{\Delta_1}\left[\Delta_1\Delta_2 - N^2 - \frac{p^2}{2}\right]C_0 + p^2 C_1 = 0 \\ p^2 C_0 + 2\frac{\kappa-1}{\kappa+1}\frac{1}{\Delta_2}\left[\Delta_1\Delta_2 - N^2 - \frac{p^2}{2}\right]C_1 = -\frac{\kappa-1}{\kappa+1}\frac{2p^2}{Ga}\sin\alpha\cdot P_p \end{cases}$$

after solving it the expressions for the transformants were found

$$\begin{aligned} W_{\alpha\beta p}(z) &= \frac{\sin\alpha}{Ga}\cdot P_p\frac{\Delta_2}{\widetilde{\Delta}}\left[-2N^2 e^{-\Delta_1 z} + (2N^2+p^2)e^{-\Delta_2 z}\right] \\ Z_{\alpha\beta p}(z) &= \frac{\sin\alpha}{Ga}\cdot P_p\frac{N^2}{\widetilde{\Delta}}\left[-2\Delta_1\Delta_2 e^{-\Delta_1 z} + (2N^2+p^2)e^{-\Delta_2 z}\right] \end{aligned} \tag{17}$$

$$\widetilde{\Delta} = 4N^4 + 4N^2p^2 + p^4 - 4N^2\Delta_1\Delta_2 \tag{18}$$

Based on formulas (9), (12), the transformants of the remaining displacements are found

$$u_{\alpha\beta p}(z) = -\frac{\alpha}{N^2}Z_{\alpha\beta p}(z),\ v_{\alpha\beta p}(z) = \frac{i\beta}{N^2}Z_{\alpha\beta p}(z) \tag{19}$$

Thus, an exact solution to the posed vector problem (13) (14) in the transform space was obtained.

2.3. Construction of the original solutions

After applying the inverse integral transformations to the solution (17), the original vertical displacement was obtained

$$W(x,y,z,t) = \frac{1}{2\pi^2}\frac{1}{Ga}\frac{1}{2\pi i}\int\limits_{l}\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty}\frac{\Delta_2}{\widetilde{\Delta}}P_p\left[-2N^2e^{-\Delta_1 z}+\right.$$
$$\left.+(2N^2+p^2)e^{-\Delta_2 z}\right]\sin\alpha\, e^{i\beta y}\sin\alpha x\, e^{pt}dp\, d\beta\, d\alpha$$

$l = (\lambda - i\infty,\ \lambda + i\infty)$

Using the parity of the integrand and applying Euler's formula, displacement is rewritten in the form

$$W(x,y,z,t) = \frac{1}{4\pi^2}\frac{1}{Ga}\frac{1}{2\pi i}\int\limits_{l}P_p\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{\Delta_2}{\widetilde{\Delta}}\left[-2N^2e^{-\Delta_1\cdot z}+\right.$$
$$\left.+(2N^2+p^2)e^{-\Delta_2\cdot z}\right]e^{i\beta y}\left[e^{-i(x-1)\alpha}-e^{-i(x+1)\alpha}\right]e^{pt}dp\, d\beta\, d\alpha$$

In order to get rid of the double integral over the parameters of the Fourier transforms, the relation connecting the Fourier and Hankel transforms [14] was used

$$\frac{1}{4\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}F\left(\sqrt{\alpha^2+\beta^2+\chi_i^2}\right)e^{-i\alpha x-i\beta y}d\alpha d\beta = \int\limits_{0}^{\infty}sF(\sqrt{s^2+\chi_i^2})\times$$
$$\times J_0(s\sqrt{x^2+y^2})ds$$

where $J_0(s)$ is the Bessel function, $\chi_1 = p,\ \chi_2 = \sqrt{\frac{\kappa-1}{\kappa+1}}p$. After simplifications, formula for displacement takes a form

$$W(x,y,z,t) = \frac{1}{\pi Ga}\frac{1}{2\pi i}\int\limits_{l}P_p\int\limits_{0}^{\infty}\frac{F(s)}{\Delta_s}\cdot s\left[J_0(s\sqrt{(x-1)^2+y^2})-\right.$$
$$\left.-J_0(s\sqrt{(x+1)^2+y^2})\right]e^{pt}ds\, dp$$

$$F(s) = \sqrt{s^2+\frac{\kappa-1}{\kappa+1}p^2}\cdot\left[-4s^2e^{-\sqrt{s^2+p^2}}+(2s^2+p^2)e^{-\sqrt{s^2+\frac{\kappa-1}{\kappa+1}p^2}}\right]$$
$$\Delta_s = 4s^4+4s^2p^2+p^4-4s^2\sqrt{s^2+p^2}\sqrt{s^2+\frac{\kappa-1}{\kappa+1}p^2}$$

Using the parity of the Bessel function $J_0(s)$, continue the integrand in an odd way to the interval $(-\infty,\ 0)$

$$W(x,y,z,t) = \frac{1}{\pi G a}\frac{1}{2\pi i}\int\limits_{l} P_p \int\limits_{-\infty}^{\infty} \frac{F(s)}{\Delta_s}\cdot s\left[J_0(s\sqrt{(x-1)^2+y^2}) - \right.$$
$$\left. -J_0(s\sqrt{(x+1)^2+y^2}\right] e^{pt} ds\, dp$$

According to the obtained solution, the displacement from the distributed over a rectangular area load can be found

$$W^{AB}(x,y,z,t) = \frac{1}{\pi G a}\frac{1}{2\pi i}\int\limits_{0}^{A}\int\limits_{-B}^{B}\int\limits_{l} P_p \int\limits_{-\infty}^{\infty} \frac{F(s)}{\Delta_s}\cdot s\left[J_0(s\sqrt{(x-a)^2+(y-b)^2}) - \right.$$
$$\left. -J_0(s\sqrt{(x+a)^2+(y-b)^2}\right] e^{pt} ds\, dp\, da\, db \quad (20)$$

Formula was written in the initial coordinate system.

2.4. Steady-state oscillation case

Suppose that the load applied across the area $0 < x < A;\ -B < y < B$ over the plane $X0Y$ changes according to the harmonic law $P(t) = e^{i\omega t}$ and $p(x,y) = P$, where P — constant intensity of the load, ω — is a natural frequency of vibrations. In this case, substituting into the constructed solution (20) $p = i\omega$, the displacement is written in the form

$$W^{AB}(x,y,z;\omega) = \frac{P}{\pi G a}\int\limits_{0}^{A}\int\limits_{-B}^{B}\int\limits_{-\infty}^{\infty} \frac{F(s;\omega)}{\Delta_{s\omega}}\cdot s\left[J_0(s\sqrt{(x-a)^2+(y-b)^2}) - \right.$$
$$\left. -J_0(s\sqrt{(x+a)^2+(y-b)^2})\right] ds\, da\, db \quad (21)$$

$$F(s;\omega) = \delta_2\left[-2s^2 e^{-\delta_1 z} + (2s^2-\omega^2)e^{-\delta_2 z}\right]$$

$$\Delta_{s\omega} = 4s^4 - 4s^2\omega^2 + \omega^4 - 4s^2\delta_1\delta_2 = (2s^2-\omega^2)^2 - 4s^2\delta_1\delta_2. \quad (22)$$

$$\delta_1 = \sqrt{s^2-\omega^2},\ \delta_2 = \sqrt{s^2 - \tfrac{\kappa-1}{\kappa+1}\omega^2} \quad (23)$$

Since the expression (23) includes the multivalued functions [3], they have to be fixed. And after making cuts, using the contour integration methods, the displacement is calculated. It is necessary that from the loaded rectangle on the quarter space's face where the load is applied, the energy is carried away to infinity by each of the two types of possible waves. These requirements make it possible to fix multivalued functions $\sqrt{s^2-\omega^2}$ и $\sqrt{s^2-\frac{\kappa-1}{\kappa+1}\omega^2}$ [3; 8]

$$\begin{aligned} &\text{when } |s|>\omega;\ |s|>\tfrac{\kappa-1}{\kappa+1}\omega:\ \delta_1=\sqrt{s^2-\omega^2};\ \delta_2=\sqrt{s^2-\tfrac{\kappa-1}{\kappa+1}\omega^2}\\ &\text{when } |s|<\omega;\ |s|<\tfrac{\kappa-1}{\kappa+1}\omega:\ \delta_1=-i\sqrt{\omega^2-s^2};\ \delta_2=-i\sqrt{\tfrac{\kappa-1}{\kappa+1}\omega^2-s^2}\end{aligned} \tag{24}$$

Damping into the environment was introduced. The energy flow must be directed away from the place where the load is applied. The root of the equation (22), [3], is the number $s=\pm k_R$ — the wavenumber related to the propagation velocity of the Rayleigh wave. The denominator has no other roots for such a fixation of δ_1 and δ_2. Going around the branch points in the corresponding loops, choosing δ_1 and δ_2 on the corresponding sections of the loop, so that the requirements (24) are satisfied. Also taking into account the residue in the Rayleigh root, the solution for plane $z=0$ is obtained

$$\begin{aligned}\frac{G}{P}W^{AB}(x,y,0;\omega) &= -\frac{2i\omega^2\sqrt{k_R^2-\frac{\kappa-1}{\kappa+1}\omega^2}}{F'(k_R)}J^{A,B}_{k_R,1}(x,y)+\\ &+\frac{2i}{\pi}\omega^2\int\limits_0^{\sqrt{\frac{\kappa-1}{\kappa+1}}\omega}\frac{s\sqrt{\frac{\kappa-1}{\kappa+1}\omega^2-s^2}}{(2s^2-\omega^2)^2+4s^2\sqrt{\omega^2-s^2}\sqrt{\frac{\kappa-1}{\kappa+1}\omega^2-s^2}}J^{A,B}_{s,1}(x,y)ds+\\ &+\frac{8i}{\pi}\omega^2\int\limits_{\sqrt{\frac{\kappa-1}{\kappa+1}}\omega}^{\omega}\frac{s^2\left(s^2-\frac{\kappa-1}{\kappa+1}\omega^2\right)\sqrt{\omega^2-s^2}}{(2s^2-\omega^2)^4+16s^4\left(s^2-\frac{\kappa-1}{\kappa+1}\omega^2\right)(\omega^2-s^2)}J^{A,B}_{s,1}(x,y)ds\end{aligned} \tag{25}$$

$$\text{Where } J^{A,B}_{s,1}(x,y)=\int\limits_0^A\int\limits_{-B}^B\left[J_0\left(s\sqrt{(x-a)^2+(y-b)^2}\right)-\right.\\ \left.-J_0\left(s\sqrt{(x+a)^2+(y-b)^2}\right)\right]da\,db \tag{26}$$

$$F'(s)=8s\left(2s^2-\omega^2\right)-\frac{4s^3\sqrt{s^2-w^2}}{\sqrt{s^2-\frac{\kappa-1}{\kappa+1}\omega^2}}-\frac{12s^3-8s\omega}{\sqrt{s^2-\omega^2}}$$

$k_R=\frac{7-\kappa}{6.84-1.12\kappa}\omega$, where the approximate formula from [3] was used.

If the formula (25) is being rewritten in terms of wavenumbers

$$k_2 = \frac{\omega}{c_2},\ k_1 = \sqrt{\frac{\kappa - 1}{\kappa + 1}}\omega = \frac{\omega}{c_1}$$

c_1 is longitudinal wave velocity; c_2 is shear wave velocity. The value of the integrand in (21) $\frac{F(s;\omega)\cdot s}{\Delta_{s\omega}}$ coincides with that one in Lamb's problem [3]. The difference with the work [8] is in the form of the function $J_{s,1}^{A,B}(x,y)$. Thus, under the assumption (11) that the functions $Z_\beta(0,z)$ and $\tilde{Z}_\beta(0,z)$ are equal to zero, the solution turned out to be practically identical to the solution of the Lamb problem.

2.5. Displacement for large values vibration frequency

For large values of frequency ω, using the expansion

$$(1-x)^{\frac{1}{2}} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 - \dots \quad x^2 \le 1$$

and based on formulas (21) – (23), a calculation formula for the displacement was obtained in the form

$$\frac{G}{P}W^{AB}(x,y,0;\omega) = -\frac{i}{\pi}\int\limits_0^\infty F(s;\omega)J_{s,1}^{A,B}(x,y)ds, \tag{27}$$

where $F(s;\omega) =$

$$\frac{\omega^3 s - \frac{1}{2}\kappa_0\omega s^3 - \frac{1}{8}\kappa_0^2\frac{s^5}{\omega} - \frac{1}{16}\kappa_0^3\frac{s^7}{\omega^3} - \frac{5}{128}\kappa_0^4\frac{s^9}{\omega^5}}{4s^4\left(\sqrt{\kappa_0} - \frac{\kappa}{\kappa-1}\right) + \omega^4\sqrt{\kappa_0} + 4s^2\omega^2\left(\sqrt{\kappa_0} - 1\right) - 4s^2\left\{\frac{s^4}{\omega^2}\kappa_1 + \frac{s^6}{\omega^4}\kappa_2 + \frac{s^8}{\omega^6}\kappa_3\right\}}$$

$$\kappa_0 = \frac{\kappa+1}{\kappa-1},\ \kappa_1 = \frac{1}{8}\kappa_0^2 + \frac{1}{4}\kappa_0,\ \kappa_2 = \frac{1}{16}\kappa_0^2 + \frac{1}{16}\kappa_0,\ \kappa_3 = \frac{1}{64}\kappa_0^2 + \frac{1}{32}\kappa_0$$

2.6. Transformation of the integral $J_{s,1}^{A,B}(x,y)$ from (26)

According to the scheme [8], consider the integral $J_{s,1}^{A,B}(x,y)$. Using the integral representation for the Bessel function [15]

$$J_0(s\sqrt{(x\mp a)^2 + (y-b)^2}) = \frac{2}{\pi}\int\limits_0^{\frac{\pi}{2}} \cos\left[s(x\mp a)\cos\psi\right]\cdot\cos\left[s(y-b)\sin\psi\right]d\psi$$

which should be substituted into formula (26). After changing the order of integration and calculating the integrals as repeated, the procedure was detailed in [2], formula (26) was rewritten in the form

$$J_{s,1}^{A,B}(x,y) = \frac{8AB}{\pi} \int\limits_{0}^{\frac{\pi}{2}} S_s^{A,B}(\psi) \sin\left[sx\cos\psi\right] \cdot \cos\left[sy\sin\psi\right] d\psi, \tag{28}$$

$$\text{where } S_s^{A,B}(\psi) = \frac{\sin\left[sB\sin\psi\right]}{sB\sin\psi} \cdot \frac{1-\cos\left[sA\cos\psi\right]}{sA\cos\psi}$$

the function $S_s^{A,B}(\psi)$ is infinitely differentiable with respect to ψ and also even, therefore, the integration path can be taken equal to $[-\pi/2,\ \pi/2]$. Subsequent change of variables $\sin\psi = \tau$ allows to rewrite (28) as

$$J_{s,1}^{A,B}(x,y) = \frac{4AB}{\pi} \int\limits_{-1}^{1} F_{s,\tau}^{A,B}(x,y) \frac{d\tau}{\sqrt{1-\tau^2}}, \tag{29}$$

$$\text{where } F_{s,\tau}^{A,B}(x,y) = \frac{\sin\left[sB\tau\right]}{sB\tau} \cdot \frac{1-\cos\left[sA\sqrt{1-\tau^2}\right]}{sA\sqrt{1-\tau^2}} \cdot \sin\left[sx\sqrt{1-\tau^2}\right] \cdot \cos\left[sy\tau\right]$$

The quadrature formula of the highest degree of accuracy [16] was applied to the integral (29)

$$J_{s,1}^{A,B}(x,y) = \frac{4AB}{N} \sum_{i=1}^{N} F_{s,\tau_i}^{A,B}(x,y) \tag{30}$$

where $\tau_i = \cos\frac{2i-1}{2N}\pi,\ i = \overline{1,N}$ are the zeros of the Chebyshev polynomial of the 1st kind.

$$F_{s,\tau_i}^{A,B}(x,y) = \frac{\sin\left[sB\tau_i\right]}{sB\tau_i} \cdot \frac{1-\cos\left[sA\sqrt{1-\tau_i^2}\right]}{sA\sqrt{1-\tau_i^2}} \cdot \sin\left[sx\sqrt{1-\tau_i^2}\right] \cdot \cos\left[sy\tau_i\right] \tag{31}$$

Substituting the expression into the displacement formula (25) and (27) the final expression was constructed

$$\frac{G}{P}W^{AB}(x,y,0;\omega) = \frac{4AB}{N}\left[-\frac{2i\omega^2\sqrt{k_R^2-\frac{\kappa-1}{\kappa+1}\omega^2}}{F'(k_R)}\sum_{i=1}^{N}F^{A,B}_{k_R,\tau_i}(x,y)+\right.$$

$$+\frac{2i}{\pi}\omega^2\sum_{i=1}^{N}\int\limits_0^{\sqrt{\frac{\kappa-1}{\kappa+1}}\omega}\frac{s\sqrt{\frac{\kappa-1}{\kappa+1}\omega^2-s^2}}{\left(2s^2-\omega^2\right)^2+4s^2\sqrt{\omega^2-s^2}\sqrt{\frac{\kappa-1}{\kappa+1}\omega^2-s^2}}F^{A,B}_{s,\tau_i}(x,y)ds+$$

$$\left.+\frac{8i}{\pi}\omega^2\sum_{i=1}^{N}\int\limits_{\sqrt{\frac{\kappa-1}{\kappa+1}}\omega}^{\omega}\frac{s^2\left(s^2-\frac{\kappa-1}{\kappa+1}\omega^2\right)\sqrt{\omega^2-s^2}}{(2s^2-\omega^2)^4+16s^4\left(s^2-\frac{\kappa-1}{\kappa+1}\omega^2\right)(\omega^2-s^2)}F^{A,B}_{s,\tau_i}(x,y)ds\right] \tag{32}$$

where $F'(s)$ is defined in (26) and $F^{A,B}_{s,\tau_i}(x,y)$ – in (31). For large values of frequency ω the formula takes the form

$$\frac{G}{P}W^{AB}(x,y,0;\omega) = -\frac{4ABi}{N\pi}\sum_{i=1}^{N}\int\limits_0^{\infty}F(s;\omega)F^{A,B}_{s,\tau_i}(x,y)ds, \tag{33}$$

where the function $F(s;\omega)$ is defined in (27)

Thus, the formula has been simplified to the calculation of single integrals of continuous functions, which is not difficult if oscillations in the near zone are of interest.

2.7. Expressions for far field displacements

The calculation of integrals in (32), (33) for large values of x and y is difficult due to the presence of an oscillating function in the integrand. To eliminate this difficulty for large values of $r=\sqrt{x^2+y^2}$, the asymptotic expressions for analyzing the far field is advisable to obtained. In the integral (28) the change of variables $x=r\cos\phi,\ y=r\sin\phi,\ \lambda=tr$ was done

$$J^{A,B}_{s,1}(r\cos\phi,r\sin\phi) = \frac{4AB}{\pi}\operatorname{Im}\left\{\int\limits_0^{\frac{\pi}{2}}S^{A,B}_s(\psi)e^{i\lambda\cos(\phi-\psi)}d\psi+\right.$$

$$\left.+\int\limits_{0}^{\frac{\pi}{2}} S_s^{A,B}(\psi)e^{i\lambda\cos(\phi+\psi)}d\psi\right\} 0\le\phi\le\frac{\pi}{2} \quad (34)$$

The stationary phase method was used for the analysis of asymptotics [8; 17], where the role of the function for the analysis of asymptotics, $f(\psi)$ is played by $\cos(\phi\mp\psi)$, and the role function $\phi(\psi)$ is an infinitely differentiable function $S_s^{A,B}(\psi)$. The first integral has a stationary point and the second has not, therefore, its contribution to the asymptotics of (34) can be neglected. The first integral in (34) can be represented as the sum

$$J_{s,1}^{A,B}(r\cos\phi, r\sin\phi) = \frac{4AB}{\pi}\,\mathrm{Im}\left(\int\limits_0^{\phi} + \int\limits_{\phi}^{\frac{\pi}{2}}\right) S_s^{A,B}(\psi)e^{i\lambda\cos(\phi-\psi)}d\psi.$$

where in the first integral the stationary point is at the end of the integration path $f'(\psi) = \frac{\partial}{\partial\psi}\cos(\psi-\phi) = 0$ for $\psi=\phi$ and $f''(\psi) = -1 < 0$, a in the second integral — at the beginning of the integration path. After application of theorems 2 and 3 [17], formula (34) was rewritten

$$J_{s,1}^{A,B}(r\cos\phi, r\sin\phi) = \frac{2AB}{\sqrt{\pi s r}}\left[\sin sr - \cos sr\right]\cdot S_s^{A,B}(\phi) + O\left(\frac{1}{r}\right)\ 0\le\phi\le\frac{\pi}{2}$$
$$S_s^{A,B}(\phi) = \frac{\sin\left[sB\sin\psi\right]}{sB\sin\psi}\cdot\frac{1-\cos\left[sA\cos\psi\right]}{sA\cos\psi} \quad (35)$$

Substitution of (35) into formulas (25) and (27) makes it possible to determine the displacement $W(x,y,0;\omega)$ in the far field $r\to\infty$. As in the work [3; 8] only the Rayleigh term makes the main contribution to the asymptotic behavior of the displacement in the far field, the highest values are achieved with the angles $\phi=0$ and $\phi=\frac{\pi}{2}$

$$J_{k_R,1}^{A,B}(r\cos\phi, r\sin\phi)\Big|_{\phi=0;\phi=\frac{\pi}{2}} = \sqrt{\frac{\pi}{2k_R}}\left(\sin k_R r - \cos k_R r\right)\times$$
$$\times\left[-\frac{\cos k_R A}{k_R A};\ \frac{\sin k_R B}{k_R B}\right] + O\left(\frac{1}{r}\right) \quad (36)$$

2.8. Discussion and numerical results

For numerical implementation, the displacement should be multiplied by $e^{i\omega t}$ and the real or imaginary part should be separated. The graphs are given for the function $\frac{G}{\rho}\,\mathrm{Im}\,W^{AB}(x,y,0;\omega)$ from (32) for values of Poisson's ratio $\mu=\frac{1}{3}$ and $\mu=\frac{1}{4}$ for frequencies $\omega=0.3;\ 1;\ 3$. For large values of frequencies formula (33) was used. Three forms of the load distribution section across the face $z=0$ were considered

1. $B=A/2$ - the load is distributed over a square;

2. $B=A$ - the load is distributed over a rectangle extended along the Oy axis;

3. $B=A/4$ - the load is distributed over a rectangle extended along the Ox axis.

To analyze the far-field $r\to\infty$, the asymptotic equalities (35), (36) were used, substituted into the expressions for the displacement (25), (27)

Comparing the graphs of vertical displacements for the same frequency $\omega=0.3$ and Poisson's ratio $\mu=1/3$ under different sections of the load distribution (Fig. 1, Fig. 2, Fig. 3), it can be seen that the maximum absolute values equal to 2.5 achieved with the shape of the section $B=A$, which corresponds to a rectangle elongated along the y-axis. At the same time, the displacement has a maximum amplitude which is approximately 2 units. In the case when the load is distributed over a rectangle elongated along the x-axis, the displacement has a minimum amplitude 0.6 and its maximum displacement is about 0.7 units.

In the case when the load is distributed over the square $B=A/2$, with an increase in the vibration frequency (Fig. 1, Fig. 4, Fig. 7), the amplitude of displacement grows. In addition, in the case when the oscillation frequency is equal to 3, negative displacements are observed, which means the lifting of the face of the quarter space. Also growing of the amplitude with increasing frequency can be seen from Fig. 2 and Fig. 5, which corresponds to the case $B=A$, where the amplitude increased from 2 units ($\omega=0.3$) to 4 units ($\omega=1$). There is also the effect of raising the edge of a quarter space due to the presence of negative amplitudes' zones (Fig. 5).

Comparing the value of vertical displacements for different values of Poisson's ratio (Fig. 5, 6), it can be seen that the behavior of the graphs is similar, but for values of $\mu=1/3$ the amplitude of oscillations is greater.

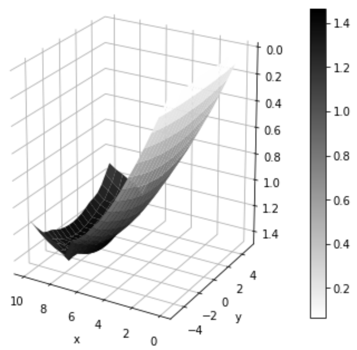

Figure 1: $B = A/2,\ \omega = 0.3,\ \mu = 1/3$

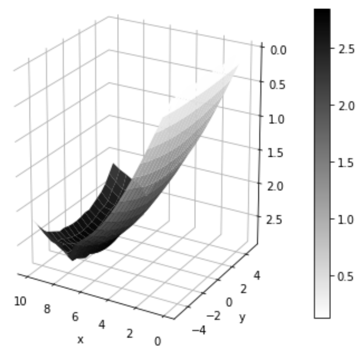

Figure 2: $B = A,\ \omega = 0.3,\ \mu = 1/3$

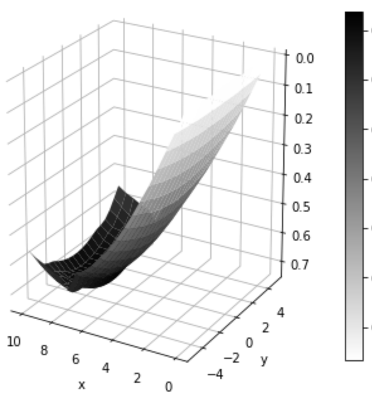

Figure 3: $B = A/4,\ \omega = 0.3,\ \mu = 1/3$

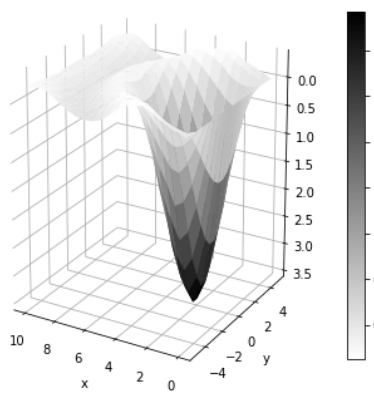

Figure 4: $B = A/2,\ \omega = 1,\ \mu = 1/3$

The vertical displacements' graphs in the remote zone of the load application area, depending on the vibration frequency with Poisson's ratio equal to 1/3 and load section $B = A$, represented in the Figure 8. As the distance from the load distribution section increases, the oscillations decay. Similar to the results for the near load zone, the maximum displacements occur in the case of the load section' shape B = A. The amplitude is greater for large values of vibration frequencies. With a decrease in the frequency of oscillations, the amplitudes are practically equal to zero.

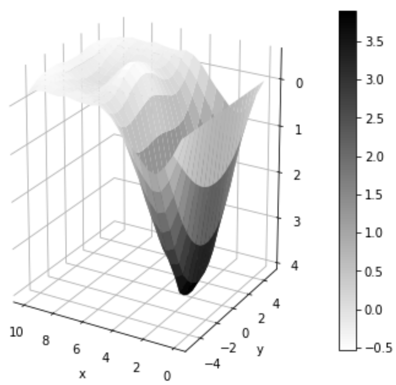

Figure 5: $B = A,\ \omega = 1,\ \mu = 1/3$

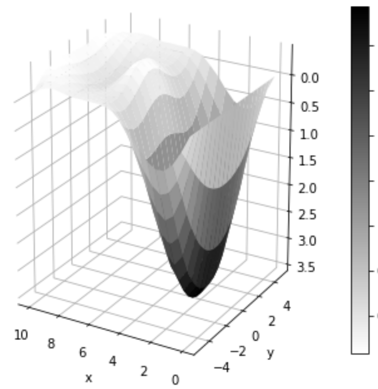

Figure 6: $B = A,\ \omega = 1,\ \mu = 1/4$

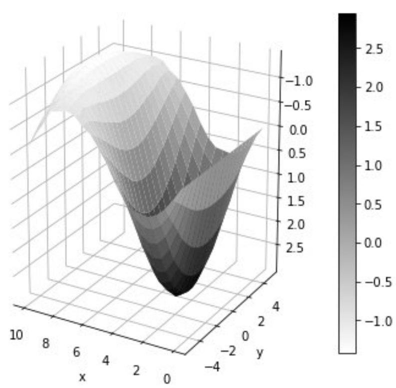

Figure 7: $B = A/2,\ \omega = 3,\ \mu = 1/3$

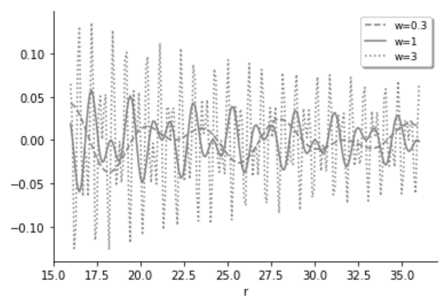

Figure 8: $B = A,\ \mu = 1/3$.

3. Conclusion

The dynamical problem's solution of the elasticity for the quarter space was derived, when one the faces is rigidly fixed and another is under the influence of the normal dynamic compressive load, applied at the initial moment of time and distributed across a rectangular section. Application of the integral transform method directly to the movement equations reduced the initial problem to the one-dimensional vector problem. The last one was solved exactly using the matrix differential calculus. The proposed approach makes it possible to obtain an exact solution of the problem in the transform's space. The case of steady state oscillations was investigated and vertical amplitude was analyzed in near loading and remote zone, for which asymptotic formulas were derived.

At the same time, it is also possible to construct and study the normal stress arising in a quarter space and compare the amplitudes of all three displacements. Using the proposed approach, the similar dynamical problem for the elastic semi-infinite layer, when different boundary conditions are set on the bottom face is under consideration.

Фесенко Г. О., Бондаренко К. С.

Динамічна задача про дію зосередженого навантаження на пружний чверть простір

Резюме

Побудовано хвильове поле пружного чверть простору, коли одну границю жорстко закріплено, а на іншій по прямокутній ділянці діє нестаціонарне нормальне стискаюче навантаження в початковий момент часу. Інтегральні перетворення Лапласа та Фур'є застосовано послідовно до рівнянь руху та до граничних умов, на відміну від традиційних підходів, коли інтегральні перетворення застосовуються до подання розв'язків через гармонічні функції. Це приводить до одновимірної векторної однорідної крайової задачі відносно невідомих трансформант переміщень. Задачу розв'язано за допомогою матричного диференціального числення. Поле вихідних переміщень знайдено після застосування обернених інтегральних перетворень. Для випадку стаціонарних коливань вказано спосіб обчислення у розв'язку квадратур в ближній зоні навантаження. Для аналізу коливань у віддаленій зоні побудовано асимптотичні формули. Досліджено амплітуду вертикальних коливань в залежності від форми ділянки навантаження, власних частот коливань та матеріалу середовища.
Ключові слова: точний розв'язок, пружний чвертьпростір, динамічне навантаження, інтегральні перетворення.

Фесенко А. А., Бондаренко К. С.

Динамическая задача о действии сосредоточенной нагрузки на упругое четверть пространства

Резюме

Построено волновое поле упругого четверть пространства, когда одна грань жестко закреплена, а на другой по прямоугольному участку действует нестационарная нормальная сжимающая нагрузка в начальный момент времени. Интегральные преобразования Лапласа и Фурье применены последовательно к уравнениям движения и граничным условиям, в отличие от традиционных подходов, когда интегральные преобразования применяются к представлениям решений через гармонические функции. Это приводит к одномерной векторной однородной краевой задаче относительно неизвестных трансформант перемещений. Задача решена с помощью матричного дифференциального исчисления. Поле исходных перемещений найдено после применения обратных интегральных преобразований. Для случая стационарных колебаний указан способ вычисления в решении квадратур в ближней зоне нагружения. Для анализа колебаний в отдаленной зоне построены асимптотические формулы. Исследована амплитуда вертикальных колебаний в зависимости от формы участка нагрузки, собственных частот колебаний и материала среды.

Ключевые слова: точное решение, упругое чтвертьпространство, динамическая нагрузка, интегральные преобразования.

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GENERATOR OF PRN'S ON THE NORM GROUP

Let p be a prime number, $d \in \mathbb{N}$, $\left(\frac{-d}{p}\right) = -1$, $m > 2$, and let E_m denotes the set of of residue classes modulo p^m over the ring of Gaussian integers in imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ with norms which are congruented with 1 modulo p^m. In present paper we establish the polynomial representations for real and imagimary parts of the powers of generating element $u+iv\sqrt{d}$ of the cyclic group E_m. These representations permit to deduce the "rooted bounds" for the exponential sum in Turan-Erdös-Koksma inequality. The new family of the sequences of pseudo-random numbers that passes the serial test on pseudorandomness was being buit.
MSC: 11L07, 11T23, 11T71, 11K45.
Key words: imaginary quadratic field, norm group, pseudorandom numbers, discrepancy.
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1. Introduction

The sequence of real numbers $\{a_n\}$, $0 \le a_n < 1$ we call the sequence of pseudorandom numbers (abbreviation, PRN's) if it is produced by deterministic generator and, being a periodical sequence, has the statistical properties such that it looks like to implementation of the sequence of random numbers with independent and uniformly distributed values on $[0, 1)$. Primary sequences of PRN's are the sequences of PRN's which generated by the congruential recursion of the type

$$y_{n+1} \equiv f(y_n, y_{n-1}, \ldots, y_{n-k+1}) \pmod{m}$$

with some initial values $y_0, y_1, \ldots, y_{k-1} \in \{0, 1, \ldots, m-1\}$, where $f(u_1, \ldots, u_k)$ is integer-valued function over $\mathbb{Z}_m^k$. Such sequences have been studied with many results (see, survey [12]).

Because it emerged that linear function $f(u) = au + b$ does not supply requirements of "affinity" to statistical independent (unpredictable) sequence (see, [10]), this motivated the creation of nonlinear congruential pseudorandom sequences having an unpredictability property.

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The generator produced by the quadratic function $f(u) = au^2 + bu + c$ satisfies to condition of “practical” unpredictability (see, [6]).

The generator associated with quadratic function $f(c)$ we call parabolical.

In 1989 J. Eichenauer and J. Lehn[4] and H. Niederreiter[13] have studied the sequences generated by the congruential relation modulo p

$$x_{n+1} = \begin{cases} ax_n^{-1} + b & if \quad x_n \neq 0, \\ b & if \quad x_n = 0. \end{cases}$$

with some coefficients $a \in \mathbb{F}_q^*$, $b \in \mathbb{F}_q$.

In the paper [18] there are investigated the analogous of inversive congruential generators, that without any increases of computational complexity of finding the elements of sequence $\{y_n\}$, have got an essential complexity for intruder's to work around the parameters of inversive or linear generator to be recovered.

The requirements to uniform distribution and unpredictability is satisfied the following inversive generator

$$y_{n+1} \equiv ay_n^{-1} + b \pmod{p^m},$$

where p is a prime number, $a, b \in \mathbb{Z}$, y_n^{-1} is a multiplicative inverse to y_n $(\text{mod } p^m)$.

The inversive generator and its generalization was being investigated by many authors (see, [1], [2], [3], [5], [6], [7], [8], [11], [15], [16], [17], [18]).

Starting out from our reasoning, we will call such inversive generator as hyperbolical.

In [19] there have been studied the statistical properties of sequences of PRN's produced by a number generator, which determines by the norm group of the ring of residue classes of modulus p^m of the ring of Gaussian integers. That generator we call circular generator.

In present paper we consider the generalization of generator from [19] and study the statistical properties of the sequences of PRN's produced by this generator.

Our main aim here is to elucidate the motivation for constructing circular generator of the sequences of PRN's with some specific properties that be faster of its usage in cryptography. Our exposition focuses on some special measures of "randomness" with respect to which "the good" sequences have been produced by using of norm group E_m. A quantive measure of uniformity of distribution of a sequence may be the so-called discrepancy. Originated from a classical problem in Diophantine approximations this concept has found applications in the analysis of PR sequences on uniformity and unpredictability. From the well-known Turan-Erdös-Koksma inequality it is evident that the main tool in estimating discrepancy is the use of bounds on exponential sums over on elements of the sequence of PRN's. This motivates a construction this paper.

Before we proceed further we will fix the notation that will be used throughout this paper.

Notation.

- Lower case Roman (respectively, Greek) letters usually denote rational (respectively, nonrational) integers of field $\mathbb{Q}$ (respectively, field $\mathbb{Q}(\sqrt{-d})$, d is a free-square natural number); in particular, m, n, k are positive integers and p is a rational prime number.

- We also define a *norm* over $\mathbb{Q}(\sqrt{-d})$ into $\mathbb{Q}$ by $N(\alpha) = a^2 + db^2$ for $\alpha = a + b\sqrt{-d}$, $a, b \in \mathbb{Q}$.

- For the sake of convenience, we suppose $d \equiv 1 \pmod 4$ and denote by G the set of integer elements of $\mathbb{Q}(\sqrt{-d})$.

- Let $\mathbb{Z}_q$ (or G_q) denotes the ring of residue classes modulo q, and $\mathbb{Z}_q^*$ (or G_q^*) denotes the multiplicative group in $\mathbb{Z}_q$ (or G_q).

- If $x \in G_q^*$, we write x^{-1} for the multiplicative inverse of x mod q, i.e. x^{-1} is an arbitrary integer of $\mathbb{Q}(\sqrt{-d})$ satisfying the condition $x \cdot x^{-1} \equiv 1 \pmod q$.

- For $a \in \mathbb{Z}$ the symbol $\left(\frac{a}{p}\right)$ denotes a symbol of Legendre.

- As usual, (a, b) stand for the greater common divisor of integer rational a and b (or, respectively, α and β in G).

- Through $\mathbb{Z}[x]$ (or $G[x]$) we denote the polynomial ring over $\mathbb{Z}$ (or G).

- For $a \in \mathbb{Z}$ ($\alpha \in G$) stand $\nu_p(a)$ (or $\nu_p(\alpha)$) if $p^{\nu(a)} | a$ and $p^{\nu(a)+1} \nmid a$.

- The fraction $\frac{a}{b}$, $(b, q) = 1$, of modulus q means as ab^{-1}, where b^{-1} is a multiplicative inverse modulo q.
- At last, $e_q(x)$ denotes $e^{2\pi i \frac{x}{q}}$.

2. Auxiliary arguments

We start by listing some previous estimates of exponential sums which will be used to establish our main results.

Let $f(x)$ be a periodic function with a period τ. For any $N \in \mathbb{N}$, $1 \le N \le \tau$, we denote

$$S_N(f) := \sum_{x=1}^{N} e^{2\pi i f(x)}$$

Lemma 1. *The following estimate*

$$|S_N(f)| \le \max_{1 \le n \le \tau} \left| \sum_{x=1}^{\tau} e^{2\pi i \left(f(x) + \frac{nx}{\tau}\right)} \right| \log \tau$$

holds.

This statement is well-known lemma about an estimate of uncomplete exponential sum by means of the complete exponential sum (see, [9]).

Lemma 2. *Let p be a prime number and let $f(x)$ be a polynomial over $\mathbb{Z}$*

$$f(x) = A_1 x + A_2 x^2 + p(A_3 x^3 + \cdots),$$

and, moreover, let $\nu_p(A_2) = \alpha > 0$, $\nu_p(A_j) > \alpha$, $j = 3, 4, \ldots$. Then we have the following estimate

$$\left| \sum_{x \in \mathbb{Z}_{p^m}} e^{2\pi i \frac{f(x)}{p^m}} \right| = \begin{cases} p^{\frac{m+\alpha}{2}} & \textit{if } \nu_p(A_1) \ge \alpha, \\ 0 & \textit{else}, \end{cases}$$

(see, [16]).

The relevant statistical properties of any sequence of the independent and uniformly distributed random numbers are, first of all, uniformity and dependence. Departures from uniformity or independency may be detected by theoretical or empirical tests. The main tools of theoretical tests for the establishment of the uniformity or dependency of the sequence $\{x_n\}$ is the s-dimensional discrepancy of the points $X_n^{(s)} = (x_n, x_{n+1}, \dots, x_{n+s-1})$, $s = 1, 2, \dots$, which defined by

$$D_N(X_0^{(s)}, X_1^{(s)}, \dots, X_{N-1}^{(s)}) := \sup_{\Delta\subset[0,1)^s} \left| \frac{A_N(\Delta)}{N} - vol(\Delta) \right|,$$

where $A_N(\Delta)$ is the number of points $X_n^{(s)}$ falling into $\Delta \subset [0,1)^s$, $vol(\Delta)$ is a volume of Δ, and the supremum is extended over all subintervals Δ of $[0,1)^s$.

If $D_N(X_0^{(s)}, X_1^{(s)}, \dots, X_{N-1}^{(s)}) \to 0$ for $N \to \infty$ we say that the sequence of PRN's $\{x_n\}$ passes the s-dimensional test on the pseudo-randomness.

The following two lemmas give the estimate for $D_N(X_0^{(s)}, X_1^{(s)}, \dots, X_{N-1}^{(s)})$.

Lemma 3. *Let $T \geq N \geq 1$ and $q \geq 2$ be integers, $\mathbf{y_k} \in \{0, 1, \dots, q-1\}^s$ for $k = 0, 1, \dots, N-1$; $\mathbf{t_k} = \frac{\mathbf{y_k}}{q} \in [0,1)^s$. Then*

$$D_N(\mathbf{t_0}, \mathbf{t_1}, \dots, \mathbf{t_{N-1}}) \leq \frac{s}{q} + \frac{1}{N} \sum_{\mathbf{h}\in C_s(q)} \sum_{h_0 \in \left(-\frac{T}{2}, \frac{T}{2}\right]} \frac{1}{r(\mathbf{h}, q) r(h_0, T)}$$
$$\times \left| \sum_{k=0}^{T} e(\mathbf{h}\cdot\mathbf{t_k} + \frac{kh_0}{T}) \right|$$

(see, [12])

Lemma 4. *The discrepancy of N arbitrary points $\mathbf{t_0}$,$\mathbf{t_1}$,…,$\mathbf{t_{N-1}} \in [0,1)^2$ satisfies*

$$D_N(\mathbf{t_0}, \mathbf{t_1}, \dots, \mathbf{t_{N-1}}) \geq \frac{1}{2(\pi+2)|h_1 h_2| N} \left| \sum_{k=0}^{N-1} e(\mathbf{h}\cdot\mathbf{t_k}) \right|$$

for any lattice point $\boldsymbol{h} = (h_1, h_2) \in \mathbb{Z}^2$ with $h_1 h_2 \neq 0$.

(It is the special version of Niederreiter result in [13]).

For integers $s \geq 1$ and $q \geq 2$, let $C_s(q)$ be the set of all nonzero lattice points $\mathbf{h} = (h_1, \dots, h_s) \in \mathbb{Z}^s$ with $-\frac{q}{2} < h_j \leq \frac{q}{2}$ for $1 \leq j \leq s$. Define for $\mathbf{h} \in C_s(q)$

$$r(h, q) = \begin{cases} 1 & if\ h = 0, \\ q \sin(\pi \frac{|h|}{q}) & if\ h \neq 0, \end{cases}$$
$$r(\mathbf{h}, q) = \prod_{j=1}^{s} r(h_j, q)$$

Lemma 5. *Let $\{\boldsymbol{Y}_n\}$ be the sequence of s-dimensional points in $(\mathbb{N}\cup\{0\})^s$ with a period τ, and $\boldsymbol{y}_n = \frac{\boldsymbol{Y}_n}{q} \in [0,1)^s$. Then for any N, $1 \le N \le \tau$, we have*

$$D_N^{(s)}(\mathbf{y_0}, \mathbf{y_1}, \ldots, \mathbf{y_{N-1}}) \le \frac{s}{q} + \frac{1}{N} \sum_{\mathbf{h}\in C_s(q)} \sum_{h_0 \in \left(-\frac{\tau}{2}, \frac{\tau}{2}\right]} \frac{1}{r(\mathbf{h}, q) r(h_0, q)} \times \left| \sum_{n=0}^{N-1} e(\mathbf{h}\cdot\mathbf{y_n} + \frac{nh_0}{\tau}) \right|,$$

where $\boldsymbol{h}\cdot\boldsymbol{y}$ denotes the inner product of $\boldsymbol{h}$ and $\boldsymbol{y}$.

Lemma 6. *Let $X_0, X_1, \ldots, X_{N-1} \in [0,1)^s$, $s \ge 1$ with discrepancy D_N^s. Then for any nonzero $\overline{h} = (h_1, \ldots, h_s) \in \mathbb{Z}^s$ we have*

$$\left| \sum_{n=0}^{N-1} e^{2\pi i \overline{h}\cdot X_n} \right| \le \frac{2}{\pi} \left(\left(\frac{\pi+1}{2} \right)^m - \frac{1}{2^m} \right) N D_N^{(s)} \prod_{j=1}^{s} \max(1, 2|h_j|),$$

where m is the number of nonzero coordinates of $\overline{h}$.

(see, [13])

Let p be a prime rational number, $\left(\frac{-d}{p}\right) = -1$. Let us denote by E_m the following subgroup of $G^*_{p^m}$

$$E_m := \{x \in G^*_{p^m} : \ N(x) \equiv \pm 1 \pmod{p^m}\}.$$

The subgroup E_m we call the norm group in $G^*_{p^m}$ of imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$.

The following lemma is constitutive for the sequence $\{x_n\}$ being investigated in our paper.

Lemma 7. *The norm group E_m is a cyclic group of order $2(p+1)p^{m-1}$. Let $u + iv$ denotes a generating element of E_m. Then exist $x_0, y_0 \in \mathbb{Z}^*_{p^m}$ such that*

$$\begin{aligned} &(u + \sqrt{-d}v)^{2(p+1)} \equiv 1 + p^2 x_0 + \sqrt{-d} p y_0, \\ &2x_0 + d y_0^2 \equiv -2p^2 x_0^2 \pmod{p^3} \end{aligned}$$

and for any $t = 4, 5, \ldots$ we have modulo p^m

$$\begin{aligned} &\Re((u+\sqrt{-d}v)^{2(p+1)t}) = A_0 + A_1 t + A_2 t^2 + \cdots \\ &\Im((u+\sqrt{-d}v)^{2(p+1)t}) = \sqrt{d}\cdot(B_0 + B_1 t + B_2 t^2 + \cdots). \end{aligned}$$

Moreover,

$$\begin{aligned}
&A_0 \equiv 1 \pmod{p^4},\ B_0 \equiv 0 \pmod{p^4},\\
&A_1 \equiv p^2x_0 + \frac{1}{2}dp^2y_0^2 \equiv -\frac{5}{2}x_0^2p^4 \pmod{p^5},\\
&B_1 \equiv py_0(1 - p^2x_0) \pmod{p^4},\\
&A_2 \equiv -\frac{5}{2}x_0^2p^2 \pmod{p^5},\\
&B_2 \equiv \frac{5}{2}p^3x_0y_0 \pmod{p^4},\\
&A_j \equiv B_j \equiv 0 \pmod{p^3},\ j = 3, 4, \ldots.
\end{aligned}$$

Proof. By virtue of the fact that the residue classes modulo p with $\left(\frac{-d}{p}\right) = -1$ generate a prime field G_p, it follows that the multiplicative group of this field is a cyclic group G_p^* and we always can yield a generating element of every group E_k of a reduced residue system modulo p^k, $k = 1, 2, \ldots$, in G.

Denote

$$(u + \sqrt{-d}v)^k = u(k) + \sqrt{-d}v(k),\ 0 \le k \le 2p + 1,$$

$$(u + \sqrt{-d}v)^{2(p+1)t+k} \equiv \sum_{j=0}^{m-1} \left(A_j(k) + \sqrt{-d}B_j(k)\right) \pmod{p^m}.$$

It is clear, that

$$A_j(k) = A_ju(k) - dB_jv(k);\ B_j(k) = A_jv(k) + B_ju(k).$$

And now, the description of group E_m is performed by an analogue of description of the norm group E_m in case of Gaussian field $\mathbb{Q}(i)$ E_m (in greater details see [14]).

Thus from Lemma (7) we infer.

Consequence 1. *For $k = 0, 1, \ldots, 2p + 1$, we have*

$$\begin{aligned}
&(u(k), p) = (v(k), p) = 1 \ if\ k \not\equiv 0 \pmod{\frac{p+1}{2}};\\
&u(0) = 1,\ v(0) = 0,\ (u(p + 1), p) = 1,\ p \| v(p + 1);\\
&u(k) \equiv 0 \pmod{p},\ (v(k), p) = 1 \ if\ k = \frac{p+1}{2} \ or\ \frac{3(p+1)}{2};\\
&u(k) = u(-k),\ v(k) = -v(-k).
\end{aligned}$$

Hence, for $k \not\equiv 0 \pmod{\frac{p+1}{2}}$ *we have*

$$
\begin{aligned}
&A_0(k) \equiv u(k),\ B_0(k) \equiv v(k) \pmod p,\\
&A_1(k) \equiv -pdy_0v(k),\ B_1(k) \equiv py_0u(k) \pmod{p^3},\\
&A_2(k) = -\frac{5}{2}x_0^2p^2u(k),\ B_2(k) \equiv -\frac{5}{2}x_0^2p^2v(k) \pmod{p^4},\\
&A_j(0) = A_j,\ B_j(0) = B_j,\ j = 3, 4, \ldots,\\
&A_0(p+1) \equiv -1,\ B_0(p+1) \equiv 0 \pmod{p^3},\\
&p^2\|A_1(p+1),\ p\|B_1(p+1),\ p^2\|A_2(p+1),\\
&p\|A_1(k),\ p^2\|B_1(k),\ p^2\|A_2(k),\ B_2(p+1) \equiv 0 \pmod{p^3},\\
&B_2(k) \equiv 0 \pmod{p^3}\ if\ k = \frac{p+1}{2}\ or\ \frac{3(p+1)}{2}.
\end{aligned}
$$

Lastly, we will make use the following sequences produced by a generating element $u + iv$ of the norm group E_m.

We select a random number $k \in \{0, 1, \ldots, 2p+1\}$ and consider the sequence $\{(u + \sqrt{-d}v)^{2(p+1)n+k}\}$, $n = 0, 1, \ldots, p^{m-1} - 1$.

Denote

$$x_n^{(k)} := \Re((u + \sqrt{-d}v)^{2(p+1)n+k}), \tag{1}$$

$$y_n^{(k)} := \Im((u + \sqrt{-d}v)^{2(p+1)n+k}). \tag{2}$$

Every sequence $\{x_n^{(k)}\}$ or $\{y_n^{(k)}\}$, $n = 0, 1, \ldots,$ has a period $\tau = p^{m-1}$. From Lemma 7 and its corollary we obtain the description of elements of these sequences as the polynomials at n. Besides, taking into account, that

$$
\begin{aligned}
&(u + \sqrt{-d}v)^{2(p+1)} = u_0 + \sqrt{-d}v_0,\\
&u_0 = 1 + p^2x_0,\ v_0 = py_0,\ (x_0, p) = (y_0, p) = 1
\end{aligned}
$$

and

$$x_n^{(k)} \equiv x_{n-1}^{(k)}u_0 - y_{n-1}^{(k)}v_0 \pmod{p^m}, \tag{3}$$

$$y_n^{(k)} \equiv x_{n-1}^{(k)}v_0 - y_{n-1}^{(k)}u_0 \pmod{p^m} \tag{4}$$

we may be achieved the representations of $x_n^{(k)}$, $y_n^{(k)}$ as the polynomials at x_0, y_0.

By virtue of the congruence $\left(x_n^{(k)}\right)^2 + d\left(y_n^{(k)}\right)^2 \equiv (-1)^k \pmod{p^m}$ and recursion (3) we call the sequences (1) and (2) as the sequences of PRN's produced by the norm group. The recursions (3), (4) we call the generators associated with the norm group E_m.

Family of sequences of PRN's produced by circular generator It is clear to see that, without restricting the generality, we can take that $d = 1$ and, hence, $p \equiv 3 \pmod 4$.

So, finally, we generate the family of the sequences of congruential PRN's which associated with the sequences $\{x_n(k)\}$ and $\{y_n(k)\}$. Depending on a select $k \in \{0, 1, \dots, 2p+1\}$ we will construct the special sequences of PRN's. We will distinguish three cases of class sequences depend upon the values of k:

(A) $k \not\equiv 0 \pmod{\frac{p+1}{2}}$;

(B) $k = 0$ or $p + 1$;

(C) $k = \frac{p+1}{2}$ or $\frac{3(p+1)}{2}$.

Firstly, we consider the class (A). The classes (B) and (C) may be consider by a similar way, but these classes have its specific.

So, for every $k \in \{0, 1, \dots, 2p+1\}$, $k \not\equiv 0 \pmod{\frac{p+1}{2}}$ we consider the sequences $\{x_n^{(k)}(t)\}$ and $\{y_n^{(k)}(t)\}$, $t = 0, 1, 2, \dots$. For such k we have

$$k \not\equiv 0 \quad \left(\mathrm{mod}\ \frac{p+1}{2}\right).$$

In these cases $(u(k), p) = (v(k), p) = 1$.

We denote

$$z_n^{(k)} := \frac{x_n^{(k)}}{1 + v_0(k) y_n^{(k)}} \pmod{p^m}, \tag{5}$$

where $v_0(k) = v(k) + p^2 v_1(k)$, $(v_1(k), p) = 1$.

This definition is correct by virtue of the fact that

$$1 + v_0(k) y_n^{(k)} \equiv 1 + v_0(k) B_0(k) \equiv 1 + d v_0^2(k) \equiv -u_0^2(k) \pmod p,$$
$$v_0(k) \sum_{j=1}^{m-1} B_j(k) n^j \equiv 0 \pmod p.$$

And hence, denoting $(u(k)^{-1})^2 = u(k)^{-2} \pmod{p^m}$, we have modulo p^m

$$z_n^{(k)} \equiv -(u(k))^{-2}(A_0(k) + A_1(k)n + \cdots)\Bigg(1 + (u(k))^{-2} v_0(k) B_1(k) n$$
$$+ u^{-2}(k)\left(v_0(k) B_2(k) n^2 + u^{-2}(k) v_0^2(k) B_1(k)\right) n^2 + \cdots\Bigg).$$

Now, after simple calculations, we get

$$z_n^{(k)} = -(u(k))^{-2} \sum_{j=0}^{M} A_j^{(k)} n^j,$$

where

$$\begin{aligned} A_1^{(k)} &= pu(k)^{-1} y_0 - py_0 v(k) u(k)^{-2}, \\ A_2^{(k)} &= v_0(k) A_0(k) B_2(k) + u(k)^{-2} v_0(k)^2 A_0(k) B_1^2(k) \\ &\quad + v_0(k) A_1(k) B_1(k) + A_2(k), \\ A_j^{(k)} &\equiv 0 \pmod{p^3}, \ j = 3, 4, \ldots. \end{aligned}$$

So, we obtain modulo p^m

$$z_n^{(k)} = F(n) \equiv (u(k))^{-1} \left[A_0^{(k)} + p^2 y_0 v_1(k) n + p^2 c_2(k) n^2 + p^3 G(n) \right], \qquad (6)$$

where

$$c_2(k) = y_0^2 \cdot (-2x_0 u^{-1}(k) v^2(k) - 10x_0^2 u^2(k) - 10x_0^2 u^{-1}(k) v(k) - u^{-3}(k) v^2(k)), \qquad (7)$$

where $G(n) \in \mathbb{Z}_{p^m}[n]$.

The relation (6) defines the representation of $z_n^{(k)}$ as the polynomial at n.

In case of (B) we consider the sequence $\{z_n^{(k)}\}$, $z_n^{(k)} = \frac{x_n^{(k)}}{y_n^{(k)}}$.

Finally, in case of (C) we let

$$z_n^{(k)} = \frac{x_n^{(k)}}{1 + y_n^{(k)}}$$

and similarly to (A) we infer the representation $z_n^{(k)}$ as polynomial at n.

This allows us to state the following theorem.

Theorem 1. *Let $h_1, h_2, j \in \mathbb{Z}$, $(h_1, h_2, p^m) = p^\ell$. Then for the sequence of PRN's $\{z_n^{(k)}\}$ the following estimate*

$$|S_j(h_1, h_2)| := \left| \sum_{n=0}^{p^{m-1}-1} e_{p^m}(h_1 z_n^{(k)} + h_2 z_{n+j}^{(k)}) \right| \le p^{\frac{m+\ell}{2}}$$

holds for every $j \in \{1, \ldots, 2p+1\}$.

Proof. Without less of generality that $(h_1, h_2, p^m) = 1$, using the relations (6) we can write for $k \not\equiv 0 \pmod{\frac{p+1}{2}}$

$$h_1 z_n^{(k)} + h_2 z_{n+j}^{(k)} \equiv$$
$$\equiv (u(k))^{-2}\Big[A_0^{(k)} + p^2((h_1 v(k) + h_2 v(k)(1+pO(j)))y_0 v(k) + 2\cdot h_2 c_2(k)(1+pO(j)))n$$
$$+p^2(h_1 c_2(k) + h_2 c_2(k)(1+pO(j)))n^2 + p^3 G_1(n)\Big] \pmod{p^m},$$

where c_2 defined in (7)

By the condition $(h_1, h_2, p^m) = 1$, it follows that the congruences

$$(h_1 v(k) + h_2 v(k)(1+pO(j)))y_0 v(k) + 2\cdot h_2 c_2(k)(1+pO(j)) \equiv 0 \pmod p$$
$$h_1 c_2(k) + h_2 c_2(k)(1+pO(j)) \equiv 0 \pmod p$$

cannot be realized simultaneously. Thus, by Lemma 2, we infer

$$|S_j(h_1, h_2)| \le \begin{cases} 0 & if \quad h_1 + h_2 \equiv 0 \pmod p, \\ p^{\frac{m}{2}} & if \quad h_1 + h_2 \not\equiv 0 \pmod p. \end{cases} \tag{8}$$

Consequence 2. *The discrepancy of the sequence* $\left\{\frac{X_n^{(s)}}{p^{m-1}}\right\}$, $s = 1, 2$, *has the following bound*

$$D_N^{(s)} \le \frac{s}{p^{m-1}} + \frac{2p^{\frac{m-1}{2}}}{N}\left(\frac{2}{\pi}\log p^m + \frac{7}{5}\right)^s, \ 0 < N \le \tau, \tag{9}$$

where $X_n^{(s)} = \left(z_n^{(k)}, \ldots, z_{n+s-1}^{(k)}\right)$.

This assertion follows from Lemma 4 and Theorem 1. Now we prove a lower estimate $D_N^{(2)}$.

Theorem 2. *Let p be a prime number, $p \equiv 3 \pmod 4$ and let $z_n^{(k)}$ defined by the relation (5), $k \not\equiv 0 \pmod{\frac{p+1}{2}}$. Then for the sequence $\{w_n^{(k)}\}$, $w_n^{(k)} = \frac{z_n^{(k)}}{p^m}$, $n = 0, 1, \ldots, \tau - 1$, we have*

$$D_\tau^{(2)}(W_0^{(k)}, W_1^{(k)}, \ldots, W_{\tau-1}^{(k)}) \ge \frac{1}{4(\pi+2)}p^{-\frac{m-1}{2}}, \tag{10}$$

where $W_n^{(k)} = (w_n^{(k)}, w_{n+1}^{(k)})$, $n = 0, 1, \ldots, \tau - 1$.

Proof. We take $h_1 = h_2 = 1$. Then by Theorem 1 with $j = 1$ and Lemma 7, we at one obtain

$$D_\tau^{(2)} \ge \frac{1}{2(\pi+2)}\tau^{-\frac{1}{2}} = \frac{1}{2(\pi+2)}p^{-\frac{m-1}{2}}.$$

(For detailed proof, see [16]).

Theorem 1 and 2 show that, in general, the upper bound is the best possible up to the logarithmic factor for circular congruential sequence $\{(w_n^{(k)}, w_{n+1}^{(k)})\}$, $n \geq 0$, defined by congruence (5) (or (10)).

3. Conclusion

In conclusion we have the following two remarks.

Remark 1. *It is straightforward to verify that all that we said in the case the sequence produced of the relation (5) also holds for the sequence produced by the congruence*

$$z_n^{(k)} \equiv u_0(k)x_n^{(k)} + v_0(k)y_n^{(k)} \pmod{p^m} \tag{11}$$

with $u_0(k) = u(k) + p^2u_1(k)$, $v_0(k) = v(k) + p^2v_1(k)$, $(u_1(k), p) = (v_1(k), p) = 1$.

Remark 2. *Relations (3), (4) make it possible to drive the representations* $x_n^{(k)}$, $y_n^{(k)}$ *and consequently* $z_n^{(k)}$ *as polynomials at* x_0, y_0. *Thus it may be well to construct non-trivial estimates of exponential sums over generating element of the norm group* E_m.

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Резюме

Нехай p — просте число, $d \in \mathbb{N}$, $\left(\frac{-d}{p}\right) = -1$, $m > 2$, і нехай E_m позначає множину класів лишків за модулем p^m над кільцем цілих гаусових чисел в уявному квадратичному полі $\mathbb{Q}(\sqrt{-d})$ з нормами, що дорівнюють 1 за модулем p^m. В даній статті ми отримуємо поліноміальні зображення для дійсної та уявної частин степенів породжуючого елементу $u + iv\sqrt{d}$ циклічної групи E_m. Ці зображення дозволяють отримати “кореневі границі” експоненційної суми в нерівності Турана-Ердьоша-Коксми. Також було побудовано нове сімейство послідовностей псевдовипадкових чисел, що проходять серіальний тест на псевдовипадковість.
Ключові слова: уявне квадратичне поле, норменна група, псевдовипадкові числа, дескріпансія.

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Генератор ПСЧ на норменой группе

Резюме

Пусть p — простое число, $d \in \mathbb{N}$, $\left(\frac{-d}{p}\right) = -1$, $m > 2$, и пусть E_m обозначает множество классов вычетов по модулю p^m над кольцом целых гауссовых чисел в мнивом квадратичном поле $\mathbb{Q}(\sqrt{-d})$ с нормами, которые сравнимы с 1 по модулю p^m. В данной статье мы получаем полиномиальные представления действительной и мномой честей степеней порождающего елемента $u + iv\sqrt{d}$ циклической группы E_m. Эти представления позволяют получить “корневые границы” экспоненциальной суммы в неравенстве Турана-Эрдёша-Коксмы. Также было построено новое симейство последовательностей псевдослучайных чисел, которые проходят сериальный тест на псевдослучайность.
Ключевые слова: мнимое квадратичное поле, норменная группа, псевдослучайные числа, дескрипансия.

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MODELING OF DEFORMATION OF THE BIMATERIAL WITH THIN NON-LINEAR INTERFACE INCLUSION

An incremental approach to solving the antiplane problem for bimaterial media with a thin, physically nonlinear inclusion placed on the materials interface is discussed. Using the jump functions method and the coupling problem of boundary values of the analytical functions method we reduce the problem to the system of singular integral equations (SSIE) on jump functions with variable coefficients allowing us to describe any quasi-static loads (monotonous or not) and their influence on the stress-strain state in the bulk. To solve the SSIE problem, an iterative analytical-numerical method is offered for various non-linear deformation models. Numerical calculations are carried out for different values of non-linearity characteristic parameters for the inclusion material. Their parameters are analyzed for a deformed body under a load of a balanced concentrated force system.

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1. Introduction

Problems related to contacts have been treated with great attention in the literature of the subject due to their practical significance. Most materials contain numerous subtle defects in the form of cracks and inclusions of various origin [4; 7; 10; 13; 14; 18; 20; 23]. The presence of these inhomogeneities in engineering materials affects or disturbs their elastic field and thus greatly influences their mechanical and physical properties. Composite materials take advantage of inclusions as reinforcements in the matrix to have superior properties not achievable by individual constituent materials. Such subtle inhomogeneities can have a complex structure, taking into account possible viscosity, plasticity, and other nonlinear effects. Considering non-linearity significantly

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complicates the process of solving the problem and requires the use of various approximate methods even for bodies of simple geometry [1; 5; 12; 18; 19].

It has been noted in the surveys [13–15; 22; 23] that for solving the problem for elastic bodies with thin inclusions it is possible to select five main approaches of analysis: **general theoretic** — to consider the inclusion of arbitrary form and then to decrease one of its sizes [6; 13–15]; **numerical** — to apply direct numerical methods [16]; **experimental** — to use experimental methods; **asymptotical** — to consider the stresses and displacements directly near the vicinity of heterogeneity and interface of materials by asymptotical methods in detail; **new theories of imperfect contact** — to develop a specific theory that would enable to solve the proper problems rather simple taking into account the effect of the small thickness of the defect [8; 9; 18; 23].

The idea of the last, one of the most productive approaches, is based on the principle of the conjugation of continua with different dimensions [18; 23]. An object is eliminated from consideration and its influence results in the appearance of jumps of temperature, heat fluxes, vectors of displacements and stresses in the matrix. Then stresses and other characteristics in an arbitrary point of solid are determined by the problem geometry, materials properties, external loading and jump functions. The mathematical model of inclusion is given as the **interaction conditions** equivalent to the conditions of imperfect contact between the matrix surfaces adjacent to inclusion.

Attempts to consider non-linearity in the problem of antiplane deformation of compressed semi-spaces with thin interfacial defects were made by various authors, including the study of sliding friction of contact bodies [3; 4; 7; 9; 21].

This article aims to develop a jump functions method and construct appropriate models of thin inclusions and layers whose material has essentially non-linear properties. Assume that the body thus loads non-uniformly, including multistage or cyclic loads.

Main Results

1. Formulation of the problem. Consider an infinite isotropic matrix consisting of two semi-spaces with the elastic shear constants G_k $(k = 1, 2)$. Here $Oxyz$ are the Cartesian coordinates and xOz is the plane of contact between half-spaces.

We'll study the stress-strain state (SSS) of the bulk section by the plane

xOy perpendicular to the direction z of its longitudinal shear. This section creates two half-planes S_k $(k = 1, 2)$ and the interface between them corresponds to the x-axis L (Fig. 1). On L along the segment $L' = [-a; a]$ there is a thin inclusion of thickness $2h \ll a$, mechanical properties of which in different directions may differ (orthotropy) and are characterized by constitutive equality of rather general non-linear form

$$\frac{\partial w^{in}}{\partial s} = \varpi_s \left(\sigma_{xz}^{in}, \sigma_{yz}^{in} \right), \quad s = \{x, y\}, \tag{1}$$

where the monotone function $\varpi_s \left(\sigma_{xz}^{in}, \sigma_{yz}^{in} \right)$ is chosen from general theoretical considerations or is some kind of approximation of empirical data relationships.

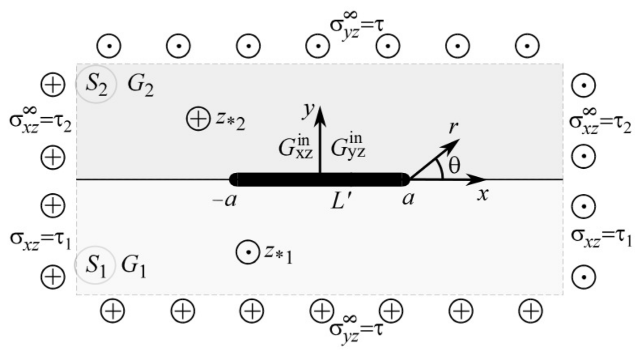

Figure 1: Geometry and load pattern of the problem

Suppose that external loading increase or decrease monotonically by arbitrary law and consist of the uniformly distributed in infinity shear stresses $\sigma_{yz}^{\infty} = \tau(t)$, $\sigma_{xz}^{\infty} = \tau_k(t)$, concentrated forces with magnitude $Q_k(t)$, screw dislocations with Burger's vector $b_k(t)$ in points $z_{*k} \in S_k$ $(k = 1, 2)$, t denotes the time as formally monotonically increasing parameter associated with load variations $(k = 1, 2)$. It should be noted that the positive direction of the forces and Burgers vectors is chosen along the axis z, in contrast to the opposite direction implicitly adopted in some studies. Since we assume the straightness of the matrix interface at infinity, we have to provide a correlation $\tau_2(t)G_1 = \tau_1(t)G_2$.

The presence of a thin inclusion in the bulk at the interface of the material is simulated by the jumps of stress tensor components and vectors of displacements on L' [18; 23]

$$\begin{aligned}
&[\sigma_{yz}]_h \cong \sigma_{yz}^- - \sigma_{yz}^+ = f_3(x,t),\\
&\left[\frac{\partial w}{\partial x}\right]_h \cong \frac{\partial w^-}{\partial x} - \frac{\partial w^+}{\partial x} = \left[\frac{\sigma_{xz}}{G}\right]_h \equiv \frac{\sigma_{xz}^-}{G_1} - \frac{\sigma_{xz}^+}{G_2} = f_6(x,t), \quad x \in L';\\
&f_3(x,t) = f_6(x,t) = 0, \quad x \notin L'.
\end{aligned} \tag{2}$$

Hereinafter the following notation is used: $[\varphi]_h = \varphi(x,-h) - \varphi(x,+h)$; $\langle\varphi\rangle_h = \varphi(x,-h) + \varphi(x,+h)$; the "+" and "−" indicators correspond to the limit values of functions at the upper and lower faces of the line L'. The contact between the upper and lower faces of inclusion and semi-spaces along the line L' and between semi-spaces along the line $L'' = L \backslash L'$ considered to be mechanically perfect

$$\begin{aligned}
&w^{in}(x,\pm h) = w_k(x,\pm h), \quad \sigma_{yz}^{in}(x,\pm h) = \sigma_{yzk}(x,\pm h) \quad (x \in L'),\\
&w_1(x,-0) = w_2(x,+0), \quad \sigma_{yz1}(x,-0) = \sigma_{yz2}(x,+0) \quad (x \in L'').
\end{aligned} \tag{3}$$

In this way, we formulate the problem of longitudinal shear in the non-bounded matrix with possible non-linear deformation of the thin interface inclusion-layer under the action of the inhomogeneous distribution of shear stresses, concentrated forces and screw dislocations. These forces can cause energy dissipation, wear, etc. in the matrix with inclusion.

2. Modeling of the presence of thin interface inclusion. The mathematical model of thin inclusion is presented in the form of so-called interaction conditions [21–23], which are equivalent to the conditions of nonperfect contact between the adjacent inclusion surfaces of the matrix. The basis of the proposed method of simulation of a thin object is the principle of volumetric integration, which consists of defining relationships describing the physical and mechanical state of the inclusion material and then considering the smallness of one of the linear dimensions of the inclusion. The main constitutive relations for the arbitrary material of inclusion are the equilibrium conditions

$$\frac{\partial \sigma_{xz}^{in}}{\partial x} + \frac{\partial \sigma_{yz}^{in}}{\partial y} = 0 \tag{4}$$

and some known stress-strain dependencies (1).

Using thin-walled proportions

$$\frac{\partial w^{in}}{\partial y}(x,h) + \frac{\partial w^{in}}{\partial y}(x,-h) \simeq \frac{w^{in}(x,h) - w^{in}(x,-h)}{h} = -\frac{[w^{in}]_h}{h}; \tag{5}$$

integrating (4) by x within $[-a, x]$ and averaging in thickness $y \in [-h, h]$

$$\frac{1}{2h}\int_{-h}^{h}\sigma_{xz}^{in}(\xi, y, t)dy \simeq \sigma_{xz}^{inAver}(x,t) = \frac{1}{2}\left\langle\sigma_{xz}^{in}\right\rangle_h(x,t) \tag{6}$$

we obtain the following form of inclusion balance conditions:

$$\frac{1}{2h}\int_{-h}^{h}\sigma_{xz}^{in}(\xi, y, t)dy - \sigma_{xz}^{in}(-a,t) - \frac{1}{2h}\int_{-a}^{x}\left[\sigma_{yz}^{in}\right]_h(\xi,t)d\xi = 0, \tag{7}$$

which, together with the relations (4), (5), fully describe the model of the thin physically non-linear inclusion, presented in SSS inclusion values. To pass to the values of the SSS of the matrix we must use the contact conditions (3). Finally, from (5), (7), and dependence (4), adding (3), we obtained the mathematical model of a thin physically non-linear inclusion [18; 23]

$$\begin{cases} -\dfrac{[w]_h}{h} = \left\langle \varpi_s\left(\sigma_{xz}, \sigma_{yz}, t\right)\right\rangle, \\ \left\langle \varpi_x^{-1}\left(\dfrac{\sigma_{xz}}{G_k}, \dfrac{\sigma_{yz}}{G_k}\right)\right\rangle_h - 2\sigma_{xz}^{in}(-a,t) - \dfrac{1}{h}\displaystyle\int_{-a}^{x}\left[\sigma_{yz}\right]_h(\xi,t)d\xi = 0. \end{cases} \tag{8}$$

The next step is to clarify the form of relationship (1). The relationship $\varpi_s(\sigma_{xz}^{in}, \sigma_{yz}^{in})$ can be given as an analytical function that reflects a specific deformation graph over the entire quasi-static load range, as well as approximate relationships based on experimental data on the measurement of the deformation properties of specific materials. It should also be noted that dependencies (1) for loading and unloading are predominantly (in the absence of ideal even non-linear elasticity) in a different form [2].

Let's consider a few partial cases of (1):

1. Let $\varpi_s\left(\sigma_{xz}^{in}, \sigma_{yz}^{in}, t\right) = 0$, $s = \{x, y\}$. This option corresponds to a completely rigid inclusion.

2. $\varpi_s\left(\sigma_{xz}^{in}, \sigma_{yz}^{in}, t\right) = const$, $s = \{x, y\}$. This option can be considered a model of ideal hard-plastic deformation of inclusion.

3. Many variants of the dependence form (1) are related to the assumption that the inclusion material is orthotropically non-linear and relation (1) can be written in a simpler form

$$\frac{\partial w^{in}}{\partial x} = \Im_x^{in}\left(\sigma_{xz}^{in}, t\right), \quad \frac{\partial w^{in}}{\partial y} = \Im_y^{in}\left(\sigma_{yz}^{in}, t\right), \tag{9}$$

or

$$\sigma_{xz}^{in} = G_x^{in}\left(\sigma_{xz}^{in}, t\right)\frac{\partial w^{in}}{\partial x}, \quad \sigma_{yz}^{in} = G_y^{in}\left(\sigma_{yz}^{in}, t\right)\frac{\partial w^{in}}{\partial y} \tag{10}$$

with variable shear modulus $G_x^{in}(\sigma_{xz}^{in})$, $G_y^{in}(\sigma_{yz}^{in})$ as specified. For example:

- Hooke's classic linear law of elasticity

$$\sigma_{xz}^{in} = G_x^{in}\frac{\partial w^{in}}{\partial x}, \quad \sigma_{yz}^{in} = G_y^{in}\frac{\partial w^{in}}{\partial y}. \tag{11}$$

- Bach-Shule plastic deformation model $\Im_s^{in}(\sigma_{sz}^{in}) = K(\sigma_{sz}^{in})^{\beta}$.
- Ramberg-Osgood deformation model,

$$\frac{\partial w^{in}}{\partial s} = A_s\sigma_{sz}^{in}\left(1 + B_s\left(\sigma_{sz}^{in}\right)^{M_s}\right), \quad s = \{x, y\}, \tag{12}$$

  where the ratio (12) coincides with the deformation model variant [2] in the case of $M_s = m_s - 1$, $A_s = 1/G_{os}$, $B_s = K_sA_s^{m_s-1}$, $s = \{x; y\}$ (G_{os}, m_s, K_s are inclusion material parameters) and when $K_s = 0.002\left(\frac{G_{0s}}{G_{0.2T}}\right)^{m_s}$, $s = \{x, y\}$ it is used to determine the "technical yield point" and the particular material parameter m_s. We note that model (12) can be considered as a non-linear ideal elastic deformation option.

- Ilyushin's model of plastic deformation

$$\Im_s^{in}(\sigma_{sz}^{in}) = \frac{\sigma_{sz}^{in}}{G_{0s}^{in}\left(1 - \omega(\sigma_{sz}^{in})\right)}. \tag{13}$$

- Classic linear elastic-plastic deformation models without hardening

$$\begin{cases} \dfrac{\partial w^{in}}{\partial s} = \dfrac{\sigma_{sz}^{in}}{G_{0s}}, & \left|\sigma_{sz}^{in}\right| < \tau_{yield}, \quad s = \{x, y\}, \\ \dfrac{\partial w^{in}}{\partial s} = \dfrac{\tau_{yield}}{G_{0s}}, & \left|\sigma_{sz}^{in}\right| \geq \tau_{yield} \end{cases} \tag{14}$$

  and with hardening

$$\begin{cases} \dfrac{\partial w^{in}}{\partial s} = \dfrac{\sigma_{sz}^{in}}{G_{0s}}, & \left|\sigma_{sz}^{in}\right| < \tau_{yield}, \quad s = \{x, y\}, \\ \dfrac{\partial w^{in}}{\partial s} = \left(\sigma_{sz}^{in} - \tau_{yield}\right)\dfrac{G_{0s} - G_{1s}}{G_{0s}G_{1s}}, & \left|\sigma_{sz}^{in}\right| \geq \tau_{yield}. \end{cases} \tag{15}$$

- Models of plastic deformation

$$\begin{aligned}
\left(\Im_s^{in}(\sigma_{sz}^{in})\right)^{-1} &= \frac{k\varepsilon_{sz}^{in}}{\sqrt{1+\left(\varepsilon_{sz}^{in}/m\right)^2}}; \\
\left(\Im_s^{in}(\sigma_{sz}^{in})\right)^{-1} &= \tau_{yield} + \frac{G_{sT}\varepsilon_{sz}^{yield}}{1-G_{sT}/G_{s0}}.
\end{aligned} \tag{16}$$

- Each deformation model is given by a function like the experimentally obtained dependence $\Im_s^{in}\left(\sigma_{sz}^{in},t\right)$ for a given material [2].

For any of the above-mentioned deformation models of the form (9), considering (10) and (3), the mathematical model of physically orthotropically non-linear thin inclusion will take the following form

$$\begin{cases} G_x^{in}(\sigma_{xz}^{in},t)\left\langle \dfrac{\partial w}{\partial x}\right\rangle_h(x,t) - 2\sigma_{xz}^{in}(-a,t) - \dfrac{1}{h}\displaystyle\int_{-a}^{x}\left[\sigma_{yz}\right]_h(\xi,t)d\xi = 0, \\ G_y^{in}(\sigma_{yz}^{in},t)\,[w]_h(x,t) + h\left\langle\sigma_{yz}\right\rangle_h(x,t), = 0. \end{cases} \tag{17}$$

Of course, the model (17) can be further complicated by considering the more complicated dependence (1), the nonperfect contact between the matrix and inclusion [8; 11; 19], the thermal load, etc. However, these complications will not be of fundamental importance for the general methodology of solving the problem.

3. The problem solution. Using the method [18; 23] to solve the problem, we can obtain dependences of stress tensor components and vector displacement derivatives on the line L of the unbounded plane at the load stage

$$\begin{aligned}
&\sigma_{yz}^{\pm}(x,t) = \mp p_k f_3(x,t) - Cg_6(x,t) + \sigma_{yz}^{0\pm}(x,t), \\
&\sigma_{xz}^{\pm}(x,t) = \mp Cf_6(x,t) + p_k g_3(x,t) + \sigma_{xz}^{0\pm}(x,t), \\
&g_r(z,t) \equiv \frac{1}{\pi}\int_{L'}\frac{f_r(x,t)dx}{x-z}, \quad p = \frac{1}{G_1+G_2}, \quad p_k = pG_k, \quad C = G_{3-k}p_k, \\
&\sigma_{yz}(z,t) + i\sigma_{xz}(z,t) = \sigma_{yz}^0(z,t) + i\sigma_{xz}^0(z,t) + ip_k g_3(z,t) - Cg_6(z,t) \\
&\qquad \left(z\in S_k;\ r=3,\,6;\ k=1,\,2\right).
\end{aligned} \tag{18}$$

Superscript "+" refers to $k=2$; "−" — $k=1$. The upper index "0" means the corresponding values in the solid body without heterogeneity under the same external loading (homogeneous solution). The following entries [21] shall continue to apply:

$$\begin{aligned} &\sigma^0_{yz}(z,t) + i\sigma^0_{xz}(z,t) = \tau(t) + i\left\{\tau_k(t) + D_k(z,t) + \right.\\ &\qquad\left. + (p_k - p_j)\,\overline{D}_k(z,t) + 2p_k D_j(z,t)\right\}, \\ &D_k(z,t) = -\frac{Q_k(t) + iG_k b_k(t)}{2\pi\,(z - z_{*k})} \quad (z \in S_k,\ k = 1,2;\ j = 3-k), \end{aligned} \tag{19}$$

Using (18), (19) and boundary conditions (3) the problem reduces (17) to a system of singular integral equations (SSIE)

$$\begin{cases} (p_2 - p_1) f_6(x,t) + 2pg_3(x,t) - \dfrac{1}{hG^{in}_x(\sigma^{in}_{xz})}\displaystyle\int_{-a}^{x} f_3(\xi,t)d\xi = F_3\left(x, G^{in}_x(\sigma^{in}_{xz}), t\right), \\ (p_2 - p_1) f_3(x,t) + 2Cg_6(x,t) - \dfrac{G^{in}_y(\sigma^{in}_{yz})}{h}\displaystyle\int_{-a}^{x} f_6(\xi,t)d\xi = F_6\left(x, G^{in}_y(\sigma^{in}_{yz}), t\right), \end{cases}$$

$$\begin{aligned} &F_3\left(x, G^{in}_x(\sigma^{in}_{xz}), t\right) = \frac{2}{G^{in}_x(\sigma^{in}_{xz})}\sigma^{in}_{xz}(-a) - \left(\sigma^0_{xz2}(x,t)/G_2 + \sigma^0_{xz1}(x,t)/G_1\right), \\ &F_6\left(x, G^{in}_y(\sigma^{in}_{yz}), t\right) = \left\langle \sigma^0_{yz}\right\rangle(x,t) - G^{in}_y(\sigma^{in}_{yz})\left(\frac{\sigma^0_{yz2}(x,t)}{G_2} + \frac{\sigma^0_{yz1}(x,t)}{G_1}\right) - \\ &\qquad\qquad - \frac{G^{in}_y(\sigma^{in}_{yz})}{h}\left[w^0\right](-a) \end{aligned} \tag{20}$$

with additional power balance conditions and the uniqueness of the displacements when traversing a thin defect

$$\int_{-a}^{a} f_3(\xi,t)d\xi = 2h\left(\sigma^{in}_{xz}(a) - \sigma^{in}_{xz}(-a)\right), \quad \int_{-a}^{a} f_6(\xi,t)d\xi = [w](a) - [w](-a). \tag{21}$$

The method [21; 23] can be used to solve SSIE (20), (21) because the characteristic part of SSIE does not depend on non-linear coefficients [17].

In general, local jump of displacement and energy dissipation are defined by expressions

$$[w](x,t) = \int_{-a}^{x} f_6(\xi,t)d\xi, \quad x \in L'; \tag{22}$$

$$W^d(t) = \int_{L'} \sigma_{yz}(x,t)\,[w](x,t)dx. \tag{23}$$

In partial cases of crack (G_x^{in}, $G_y^{in} \to 0$) or rigid inclusion (G_x^{in}, $G_y^{in} \to \infty$), SSIE (20) has analytical solutions that correspond to known results [21; 23]. In case when the materials of the semi-spaces are identical ($G_1 = G_2 = G$) SSIE (20) is simplified to two independent SIE:

$$\begin{aligned} \frac{1}{G} g_3(x,t) - \frac{1}{h G_x^{in}(\sigma_{xz}^{in},t)} \int_{-a}^{x} f_3(\xi,t) d\xi = F_3\left(x, G_x^{in}(\sigma_{xz}^{in},t), t\right), \\ G g_6(x,t) - \frac{G_y^{in}(\sigma_{yz}^{in},t)}{h} \int_{-a}^{x} f_6(\xi,t) d\xi = F_6\left(x, G_y^{in}(\sigma_{yz}^{in},t), t\right); \end{aligned} \tag{24}$$

A more detailed analysis of the solution of the problem will be carried out for the partial case (24) equality of elastic characteristics of semi-spaces. As a result of the above-mentioned method [21; 23], we use the decomposition of jump functions into a series of Chebyshev polynomials

$$f_r\left(\frac{x}{a}, t\right) = \frac{a}{\sqrt{a^2 - x^2}} \sum_{j=0}^{n} B_j^r(t) T_j\left(\frac{x}{a}\right), \quad (r = 3, 6). \tag{25}$$

Using known integrals, we obtain

$$\int_{-a}^{x} f_r(\xi,t) d\xi = \left(\frac{\pi}{2} + \arcsin\frac{x}{a}\right) a B_0^r(t) - \sqrt{a^2 - x^2} \sum_{j=1}^{n} \frac{1}{j} B_j^r(t) U_{j-1}\left(\frac{x}{a}\right),$$

$$g_r\left(\frac{x}{a}, t\right) = \sum_{j=1}^{n} B_j^r(t) U_{n-1}\left(\frac{x}{a}\right), \quad \int_{-a}^{a} f_r(\xi,t) d\xi = \pi a B_0^r(t). \tag{26}$$

Next, after transforming (24) into a dimensionless form, using (25)–(26) in the set of points $x_m = \cos\frac{m\pi}{n+1}$ $(m = \overline{1,n})$ generates two independent linear algebraic equations (SLAE) of orders $n+1$ for unknown items B_j^r $(r = 3, 6;$ $j = \overline{0,n})$

$$\begin{cases} \sum_{j=0}^{n} \chi_{mj}^3 \left(x_m, \tilde{G}_{xm}^{in}\right) B_j^3 = \tilde{F}_3\left(x_m, \tilde{G}_{xm}^{in}\right), \\ B_0^3 = 2\tilde{h}\left(\sigma_{xz}^{in}(a) - \sigma_{xz}^{in}(-a)\right)/G_{av}, \end{cases} \tag{27}$$

$$\begin{cases} \sum_{j=0}^{n} \chi_{mj}^6 \left(x_m, \tilde{G}_{ym}^{in}\right) B_j^6 = \tilde{F}_6\left(x_m, \tilde{G}_{ym}^{in}\right), \\ B_0^6 = [\tilde{w}](a) - [\tilde{w}](-a), \end{cases} \quad m = \overline{1,n}. \tag{28}$$

where the notations are used

$$\chi^3_{jm} = -\delta_{0j}\frac{\gamma_m}{\tilde{G}^{in}_{xm}} + (1-\delta_{0j})\left(\frac{\mu_{jm}}{\tilde{G}^{in}_{xm}} + 2\tilde{p}\right)\rho_{jm},$$
$$\chi^6_{jm} = -\delta_{0j}\tilde{G}^{in}_{ym}\gamma_m + (1-\delta_{0j})\left(\tilde{G}^{in}_{ym}\mu_{jm} + 2\tilde{C}\right)\rho_{jm},$$
$$\mu_{jm} = \frac{\sqrt{1-x_m^2}}{j\tilde{h}}, \quad \gamma_m = \frac{\pi}{\tilde{h}}\left(1 - \frac{m}{n+1}\right), \quad \rho_{jm} = U_{j-1}(x_m),$$
$$\tilde{G}^{in}_{sm} = \tilde{G}^{in}_{s}(x_m), \quad s = \{x, y\},$$
$$\tilde{x} = x/a, \quad \tilde{h} = h/a, \quad \tilde{y} = y/a, \quad \tilde{G}^{in}_x = G^{in}_x/G_{av}, \quad \tilde{G}^{in}_y = G^{in}_y/G_{av},$$
$$\tilde{f}_3 = G_{av}f_3, \quad \tilde{f}_6 = f_6, \quad \tilde{F}_3 = F_3/G_{av}, \quad \tilde{F}_6 = F_6/G_{av}, \quad \tilde{p} = G_{av}p$$
$$\tilde{C} = C/G_{av}, \quad G_{av} = \left\{\sqrt{G_1G_2}, \max\left(G_1, G_2\right), \tau, \tau_{yield}, Q/\pi a\right\},$$

δ_{0j} is Kronecker symbol.

Dependence $G^{in}_x(\sigma^{in}_{xz}, t)$, $G^{in}_y(\sigma^{in}_{yz}, t)$ on the current SSS causes serious calculation difficulties due to its variability along L. Therefore, to take this effect into account, we can propose the following iterative strategy for solving the problem.

Let's mark $\left(G^{in}_{sm}\left(\sigma^{in}_{sz}(x_m), t\right)\right)^k$, $s = \{x, y\}$ is a dependent shear module at collocation points x_m ($m = \overline{1, n}$) for the appropriate number of approximations k. At the initial moment (zero approximation), the values $\left(G^{in}_{sm}\right)^0(0, 0)$ are selected as equal to the initial point of the loading process G^{in}_{x0}, G^{in}_{y0} (according to the deformation diagram) in the absence of a residual SSS. These values are the same at all points of the collocation x_m ($m = \overline{1, n}$).

1. The external loading of the body starts with a relatively small value of the parameter τ or Q for the selected loading scheme (the first loading step in time $t_{(1)}$ is completed). Then we solve the SLAE (27)–(28). The obtained values $\left(B^r_j\right)^k$ ($r = 3, 6$; $j = \overline{0, n}$) are replaced into the relations (25) - (26), and then in (18), calculating the stresses and deformations in each of the collocation points

$$\begin{aligned}\sigma^{in}_{yz}(\tilde{x}, t) &= \left\langle\sigma^0_{yz}(\tilde{x}, t)\right\rangle/G_{av} - (p_2 - p_1)\tilde{f}_3(\tilde{x}, t) - 2\tilde{C}\tilde{g}_6(\tilde{x}, t),\\ \sigma^{in}_{xz}(\tilde{x}, t) &= 2\sigma^{in}_{xz}(-a)/G_{av} + \int_{-1}^{\tilde{x}}\tilde{f}_3(\xi, t)d\xi/\tilde{h}.\end{aligned} \tag{29}$$

$$\left\langle \frac{\partial w}{\partial x} \right\rangle (\tilde{x}, t) = \left\langle \frac{\sigma^0_{xz}}{G_k} \right\rangle (\tilde{x}, t) + (p_2 - p_1)\tilde{f}_6(\tilde{x}, t) + 2\tilde{p}\tilde{g}_3(\tilde{x}, t),$$
$$\left\langle \frac{\partial w}{\partial y} \right\rangle (\tilde{x}, t) = \frac{[w](-a)}{h} - \frac{1}{h} \int_{-a}^{x} \tilde{f}_6(\xi, t) d\xi + \left\langle \frac{\sigma^0_{yz}}{G_k} \right\rangle (\tilde{x}, t).$$

2. Next, we check that dependence (10) is performed at each collocation point with the specified accuracy, i.e. that the value of the module $G^{in}_{sm}(\sigma^{in}_{sz})$ corresponds to the stress $\sigma^{in}_{sz}(x_m)$ or deformation $\frac{\partial w}{\partial s}^{in}(x_m)$ level obtained by the given deformation graph. If the specified accuracy meets the requirements, we determine the current values of the modules $G^{in}_{xm}(\sigma^{in}_{xz}, t)$, $G^{in}_{ym}(\sigma^{in}_{yz}, t)$ for each collocation point and proceed to the next loading step (on item 1). If not, we repeat the calculation by replacing in SLAE (27)–(28) the module values $G^{in}_{sm}\left(\sigma^{in}_{sz}(x_m, (G^{in}_{sm})^{k-1})\right)$ at each collocation point obtained at the previous approximation, thus minimizing the deviation of the calculated module from that specified in (10). The process is convergent. Once sufficient accuracy has been reached, we return to item 1, continuing with the loading. The values obtained in the first (initial) step of the SSS matrix will affect the residuals in the second step (loading or unloading).

Using (5), (18), (25), (26) the expression of energy dissipation at the load stage take a discrete dimensionless form

$$W^d(t) = \int_{L'} \sigma_{yz}(x, t)[w](x, t)dx = 0.5\tilde{h}G_{av}a^2\tilde{W}^d(t),$$
$$\tilde{W}^d(t) = 0.5 \sum_{m=1}^{n} \left\langle \tilde{\sigma}_{yz}(x_m, t) \right\rangle [\tilde{w}] (x_m). \tag{30}$$

4. Numerical analysis and discussion. Numerical analysis of the solution of the problem is made for a partial case of an equality of elastic characteristics of half-spaces ($G_1 = G_2 = G$) under a gradual alternating load with concentrated forces $\tilde{Q} = Q/aG_{av}$ ($\tilde{Q}_2 = -\tilde{Q}_1 = \tilde{Q}$, $\tilde{z}_{*2} = -\tilde{z}_{*1} = i\tilde{d}$) in the global load as a result of the change $\tilde{Q}$: $[0 \div 10.0]$.

Fig. 3 shows a comparison of the results using the elastic-plastic deformation law (14)–(15) (dashed line, Fig. 2) and Hooke's deformation law (13) (solid line, Fig. 2).

According to the acute change like deformation in the plastic inclusion area (Fig. 3), the change in energy dissipation rate depends on the difference

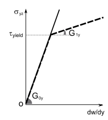

Figure 2: The elastic-plastic deformations law (14)–(15) uniaxial stress-strain diagram for inclusion material

between the diagrams (14)–(15) of the modules G_{0y}, G_{1y} (Fig. 4) and is much more pronounced when the points of force application are closer to the inclusion axis.

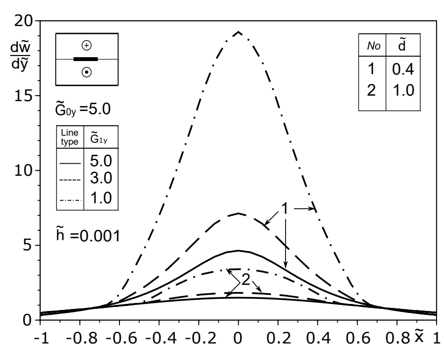

Figure 3: Distribution of elastic-plastic deformations (dash, dot-dash) of the inclusion during loading in comparison with linear elastic (Hooke's law, solid line) deformations

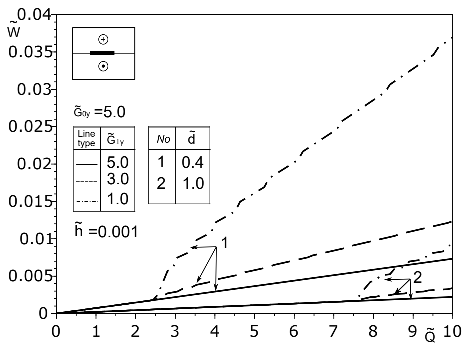

Figure 4: Dependence of change in energy dissipation rate on the difference between the modules in the G_{0y}, G_{1y} range (14)–(15) and a distance $\tilde{d}$ of the forces application points from the inclusion axis

2. Conclusion

A method of modeling of the thin inclusion with any non-linear physical-mechanical properties of the general form has been developed. This allowed solving the problem of a longitudinal shear of the matrix containing such a thin inclusion-layer at the component's boundary. SLAE with variable coefficients-functions is constructed by the methods of coupling the limit values of analytical functions and jump functions. This makes it possible to describe any way of changing the quasi-static load (monotonous or not) and its effect on the SSS in the body with inhomogeneity based on an incremental approach. For the numerical solution of the system, a convergent iterative analytical-numerical method has been proposed. Calculation formulas for deformations, SSIF, and energy dissipation are constructed. Numerical analysis of the inclusion material is subject to the linear law of elastic-plastic deformation. It has been found that the rate of energy dissipation increased significantly when plastic deformations started under load and are much more pronounced when the points of force application are closer to the inclusion axis.

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Піскозуб Й. З., Сулим Г. Т.

Моделювання деформації біматеріалу з тонким механічно нелінійним міжфазним включенням

Резюме

Обговорюється інкрементальний підхід до вирішення антиплоскої задачі для біматеріального середовища з тонким, фізично нелінійним міжфазним включенням. Використовуючи методи функцій стрибка та задачі спряження граничних значень аналітичних функцій зводимо задачу до системи сингулярних інтегральних рівнянь (ССІР) зі змінними коефіцієнтами, що дозволяють описувати будь-які квазістатичні навантаження (монотонні і немонотонні), а також їх вплив на напружено-деформований стан середовища. Для вирішення ССІР пропонується ітеративний аналітико-числовий метод для різних моделей нелінійного деформування. Виконуються чисельні розрахунки для різних значень параметрів нелінійності, що характеризують матеріал включення. Їх параметри аналізуються для тіла, що деформується під навантаженням збалансованої системи зосереджених зусиль. Здійснено числові розрахунки для різних значень параметрів нелінійності механічних характеристик матеріалу включення. Досліджено їх вплив на напружено-деформований стан матриці навантаженої збалансованою системою зосереджених сил.

Ключові слова: нелінійна пружність, тонке включення, розсіювання енергії, коефіцієнт інтенсивності напружень, антиплоскої деформація, поздовжній зсув, біматеріал, функції стрибка.

Пискозуб Й. З., Сулим Г. Т.

Моделирование деформации биматериала с тонким механически нелинейным межфазным включением

Резюме

Обсуждается инкрементальный подход к решению антиплоской задачи для биматериальной среды с тонким, физически нелинейным межфазным включением. Используя методы функций скачка и задачи сопряжения граничных значений аналитических функций, мы сводим задачу к системе сингулярных интегральных уравнений (ССИУ) с переменными коэффициентами, позволяющими описывать любые квазистатические нагрузки (монотонные и немонотонные), а также их влияние на напряженно-деформированное состояние среды. Для решения ССИУ предлагается итеративный аналитико-числовой метод для различных моделей нелинейного деформирования. Выполняются численные расчеты для различных значений параметров нелинейности, характеризующих материал включения. Их параметры анализируются для деформируемого тела под нагрузкой сбалансированной системы сосредоточенных усилий. Осуществлено числовые расчеты для различных значений параметров нелинейности упругих характеристик материала включения. Исследовано их влияние на напряженно-деформированное состояние матрицы нагружаемой сбалансированной системой сосредоточенных сил.

Ключевые слова: нелинейная упругость, тонкое включение, рассеяние энергии, коэффициент интенсивности напряжений, антиплоская деформация, продольный сдвиг, биматериал, функции скачка.

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TRIANGLE CONICS AND CUBICS

This is a paper about triangle cubics and conics in classical geometry with elements of projective geometry. In recent years, N.J. Wildberger has actively dealt with this topic using an algebraic perspective. Triangle conics were also studied in detail by H.M. Cundy and C.F. Parry recently. The main task of the article was to develop an algorithm for creating curves, which pass through triangle centers. During the research, it was noticed that some different triangle centers in distinct triangles coincide. The simplest example: an incenter in a base triangle is an orthocenter in an excentral triangle. This was the key for creating an algorithm. Indeed, we can match points belonging to one curve (base curve) with other points of another triangle. Therefore, we get a new intersting geometrical object. During the research were derived number of new triangle conics and cubics, were considered their properties in Euclidian space. In addition, was discussed corollaries of the obtained theorems in projective geometry, what proves that all of the descovered results could be transfered to the projeticve plane.

MSC: 51A05, 51A20,14H50, 14H52.

Key words: triangle cubics, conics, curves, projective geometry, Euclidian space.

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1. Introduction

Centers of triangle and central triangles were studied by Clark Kimberling [5]. We consider curves which pase throw incenter in a base triangle is an orthocenter in an excentral triangle and other triangle curves. Also we studied some properties of Jerabek hyperbola for the mid-arc triangle.

2. Main result

Firstly, we will consider excentral triangle. Correspondence between points of the excentral and base triangles will give us significant results in developing new triangle curves. Below you may observe correspondence table between

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points of the base and excentral triangles. Main analytical results of this paper belongs to Skuratovskii R. V.

Base triangle	**Excentral triangle**
I (incenter)	H (orthocenter)
O (circumcenter)	E (nine-point center)
Be (Bevan point)	O (circumcenter)
Mi (mittenpunkt)	Sy (Lemoine point)
$Mi\,'$ (isogonal conjugate of the mittenpunkt with respect to the base triangle)	M (centroid)
Sp (Speaker point)	Ta (Taylor point)
Sy (Lemoine point)	$Sy(H_1H_2H_3)$ (Lemoine point of the orthic triangle)
Mi'' (isogonal conjugate point of the mittenpunkt with respect to the excetntral triangle)	GOT (homothetic center k of the orthic and tangent triangles)

Table 1: Correspondence table between points of the base and excentral triangles

All of the above facts could be easily proved by basic principles of classiical geometry [5]. Hence, we may apply derived results for creating new triangle cubics and conics. Firstly Jerabek hyperbola was considered.

Definition 1. Jerabek hyperbola is a curve which passes through verticies of trianlge, circumcenter, orthocenter, Lemoine point, isogonal conjugate of the de Longchamps point [6].

We may observe that Jerabek hyperbola for the extcentral tringle has number of points which corresponde to other ones in the base triangle. The study of such matches gave us significant results 2.

Therefore, we got new triangle hyperbola 1, which passes through cneters of the excircles, Bevan point, incenter, mittenpunkt and de Longchamps point. It is still rectangular as Jerabek hyperbola is. Known fact about Jerabek hypperboal is that its center is center of the Euler circle. However, Euler circle of

Jerabek hyperbola for excentral triangle	**New hyperbola for the base triangle**
A, B, C (vertices of the base tringle)	I_1, I_2, I_3 (excenters)
O (circumcenter)	Be (Bevan point)
H (orthocenter)	I (incenter)
Sy (Lemoine point)	Mi (mittenpunkt)
L' (isogonal conjugate of the de Longchamps point)	L (de Longchamps point)

Table 2: Mtaching points for Jerabek hyperbola

the excentral trinangle is circumscribed circle for the base triangle. In addition, Jerabek Hyperbola is isogonaly conjugate to the Euler line. Meanwhile, Euler line for the excentral triangle is line (I (incenter), O (circumcenter), Be (Bevan point), Mi' (isogonal conjugate for the mittenpunkt with respect to the extriangle),Mi''(isogonal conjugate for the mittenpunkt with respet to the base triangle)). Therefore, we can conclude, that our new hyperbola is isogonaly conjugate to the line (I, O, Be, Mi', Mi'') and its center is the circumcenter.

Theorem 1. *New hyperbola passes though excenters, Bevan point, incenter, mittenpunkt, de Longchaps point. It is isogonal conjugate to the line* (I, O, Be, Mi', Mi'')*, and its center is circumcircle.*

Similarly we studied Thomson cubic for the base trianlge and matched its points with ones in the excetral trianlge.

Definition 2. Thomson cubic is a curve that passes through verticies of the trianlge, middles of triangle sides, centers of the excircles, incenter, centroid, circumcenter, Lemoine point,mittenpunkt, isogonal conjugate of the mittenpunkt [5].

By apllying correspondence table between points of the base and excentral triangles 1 to the points of Thomson cubic we obtain a new triangle cubic 3.

According to the table, we got new cubic.

Theorem 2. *New triangle cubic 2 passes through vertices of the triangle, bases of the altitudes, middles of the triangle sides in the orthic triangle, orthocenter,*

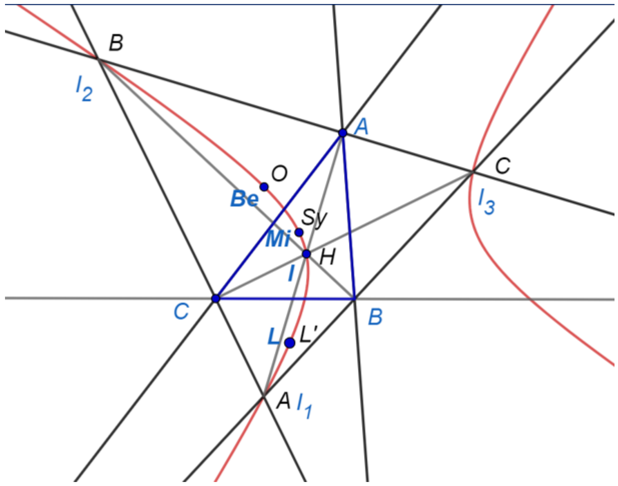

Figure 1: New trianlge conic based on Jerabek hyperbola for the excentral triangle

Euler point, centroid in orthic triangle, Lemoine point, and gomotetic center of the orthic and tangent triangles.

Analogically was derived new trianlge cubic based on the Darboux cubic and correspondence of its points with triangle centers in the excentral triangle.

Definition 3. Darboux cubic is a curve that passes through vertices of the triangle, centers of the excircles, incenter, circumcenter, Bevan point [2].

Trianlge centers of the Darboux cubic in the base triangle were matched with points in the excentral triangle.

Theorem 3. *Therefore, we got new cubic 3, which passes through vertices of the triangle, bases of the altitudes, orthocenter, Euler center, and circumcenter.*

The discussed above results were obtained from considering excentral triangle, its triangle centers and correspondence between points in the excentral and basic triangle. As a result, were derived three new triangle curves, which were not discovered before. However, to get wider results were applied the same idea to other triangles. Namely was consider medial triangle.

Definition 4. Medial triangle is a triangle with vertices in the middles of the base traingle sides.

Thomson cubic for the base triangle	**New cubic for the excentral triangle**
A, B, C (vertices of the base trianlge)	H_1, H_2, H_3 (bases of the altitudes)
M_a, M_b, M_c (middles of the base triangle sides)	M_{ha}, M_{hb}, M_{hc} (midles of orthic triangle's sies)
I_1, I_2, I_3 (excenters)	A, B, C (vertices)
I (incenter)	H (orthocenter)
M (centroid)	$M(H_1H_2H_3)$ (centroid in the orthic triangle)
O (circumcenter)	E (nine-point center)
Sy (Lemoine point)	$Sy(H_1H_2H_3)$ (Lemoine point of the orthic triangle)
Mi (mittenpunkt)	Sy (Lemoine point)
Mi'' (isogonal conjugate of the mittenpunkt with respet to the excentral triangle)	GOT (gomotetic center of the orthic anf tngent triangles)

Table 3: Mtaching points for Thomson cubic

In the same way as before, was proven the fact that some points in the medial triangle match with some points in the base triangle5. Proof of the mentioned facts relies on the patterns of the Euclidean geometry, some of the correspondence were proved before [5].

Definition 5. Yff hyperbola is a triangle curve wich passes through centroid, orthocenter, circumcenter, and Euler center.

We have considered Yff hyperbola for the base triangle and matched its point with triangle centers of the medial triangle, applying correspondence table5.

We got new conic, which has vertices in the centroid and de Longchaps point, focus in the orthocenter. Directix of the Yff hyperbola is perpendicular to the Euler line and passes through center of the Euler circle. Euler line for the medial and base triangles coincide. However, center of the Euler circle of the base triangle is circumcenter for the medial. Therefore, directrix of the new

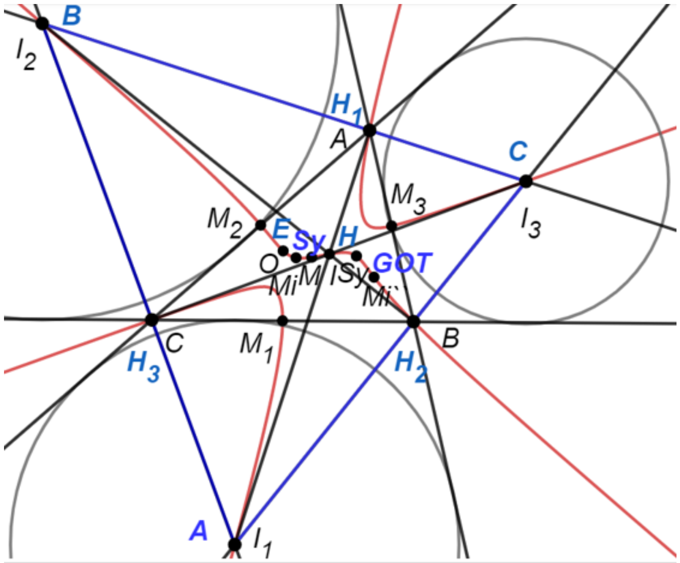

Figure 2: New trianlge conic based on Thomson hyperbola for the excentral triangle

hyperbola is perpendicular to the Euler line and passes through circumcenter.

Theorem 4. *New conic 4 is a curve that has vertices in the centroid and de Longchaps point, focus in the orthocenter. Directrix of the new hyperbola is perpendicular to the Euler line and passes through circumcenter.*

Let‘s go further in our research and create more triangle cubics with the help of correspondence between triangle centers in medial and base triangles.

We obsereved the transformation of the Darboux cubic and under the correspondence. It led us to a new cubic.

Theorem 5. *We got new cubic 5, which passes through Speaker point, center of the Euler circle, circumcenter, orthocenter, complementary conjugate of the orthocenter, middles of the triangle side, and antipodes of the medial triangle.*

Simiraly, we take Lucas cubic for the base cubic and match it with points of the medial triangle.

Definition 6. Lucas cubic is such curve which passes through triangle vertices, orthocenter, Gergone point, centroid, Nagel point, Lemoine point of the anticomplementary tringle, and vertices of th anticomplementary triangle.

Darboux cubic for the base triangle	**New cubic for the excentral triangle**
A, B, C (vertices of the base triangle)	H_1, H_2, H_3 (bases of the altitudes)
I_1, I_2, I_3 (excenters)	A, B, C (vertices)
I (incenter)	H (orthocenter)
O (circumcenter)	E (nine-point center)
Be (Bevan point)	O (circumcenter)

Table 4: Mtaching points for Darboux cubic

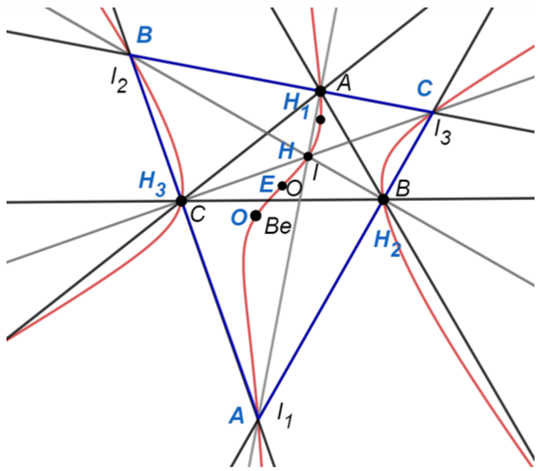

Figure 3: New trianlge conic based on Darboux cubic for the base triangle in correspondence with excentral trianlge

We make the correspondence between points of the Lucas cubic in the medial triangle with triangle centers in the base triangle. As a result we obtain the following table.

Theorem 6. *New cubic 6 which passes through Lemoine point, centroid, circumcenter, mittenpunkt, incenter, orthocenter, triangle vertices, and middles of the triangle sides.*

Therefore, while applying correspondence method to the medial triangle we derived one new conic and two new cubics. In addition, we obsereved Euler and mid-arc triangles as correspondence base.

Points in the medial triangle	**Point in the base triangle**
I (incenter)	Sp (Speaker point)
M (centroid)	M (centroid)
O (circumcenter)	E (nine-point center)
H (orthocenter)	O (circumcenter)
L (de Longchamps point)	H (orthocenter)
Be (Bevan point)	$Be(M_1M_2M_3)$ (Bevan point of the medial triangle)
Na (Nagel point)	I (incenter)
G (Gergonne point)	Mi (mittenpunkt)
Sy_A (Lemoine point of the anticomplemetary triangle)	Sy (Lemoine point)
B_3 (third Brocard point)	M_B (Brocard midpoint)

Table 5: Correspondence table between points of the medial and base triangles

Yff hyperbola for the base triangle	**New hyperbola for the medial traingle**
M (centroid)	M (centroid)
H (orthocenter)	L (de Longchaps point)
O (circumcenter)	H (orthocenter)
E (nine-point center)	O (circumcenter)

Table 6: Mtaching points for the Yff hyperbola and medial triangle

Definition 7. Euler triangle is triangle with vertices in the intersection points of the triangle altitudes and nine-point circle.

Definition 8. Mid-arc triangle is a triangle with vertices in the middles of the arcs of the circumcircle.

Let's firstly consider correspondence of points between Euler and base triangles.

According to the above table we have buil the correspondence between points of the Darboux cubic for the Euler triangle and triangle centers of the base triange.

Theorem 7. *New cubic 7 passes through circumcenter, orthocenter, Euler cen-*

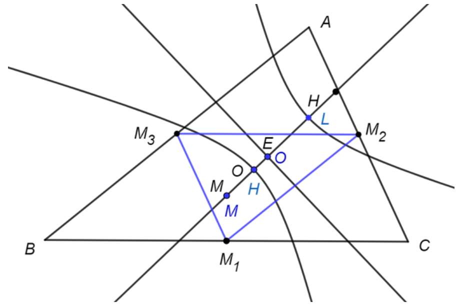

Figure 4: New conic based on Yff hyperbola

Darboux cubic for the medial triangle	**New cubic for the base triangle**
A, B, C (triangle vertices)	M_1, M_2, M_3 (middles of the triangle sides)
A_1, B_2, C_3 (antipods of the triangle)	Antipods of the medial triangle
I (incenter)	Sp (Speaker point)
O (circumcenter)	E (nine-point center)
H (orthocenter)	O (circumcenter)
L (de Longchamps point)	H (orthocenter)
L' (isogonal conjugate of the de Longchaps point)	H_A (complementary conjugate of the orthocnter)

Table 7: Mtaching points for the Darboux cubic and medial triangle

ter, midpoint of the incenter and the orthocenter, vertices of the Euler triangle and middles of the triangle sides.

Finally, we observe correspondence of traingle centers between mid-arc and base trinalge.

Based on the correspondence between point sof the mid-arc and base triangles we discovered new cubic which is based on Jerabek hyperbola.

Theorem 8. *New conic 8 is rectangular and passes through circumcenter, incenter, midpoint of mittenpunkt and incenter, Schiffler point, and isogonaly conjugate point to the Bevan point.*

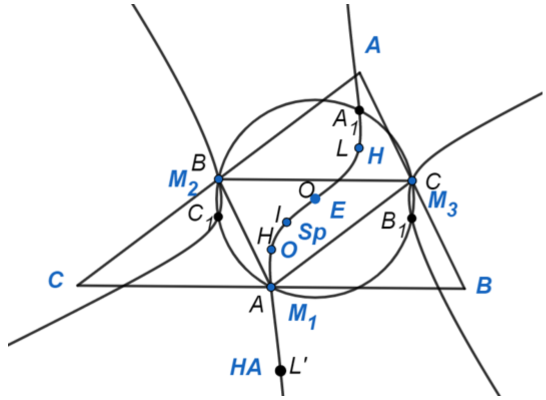

Figure 5: New cubic based on Darboux cubic for the medial triangle

Lucas cubic for the medial triangle	**New cubic for the base triangle**
Sy_A (Lemoine point of the anticomplementary triangle)	Si (Lemoine point)
M (centroid)	M (centroid)
H (orthocenter)	O (circumcenter)
Ge (Gergonne point)	Mi (mittenpunkt)
Na (Nagel point)	I (incenter)
L (de Longchamps point)	H (orthocenter)

Table 8: Mtaching points for the Lucas cubic and medial triangle

Therefore, during the research of the triangle curves were derived three new triangle conics nad five new triangle cubics. This is a significant result and leaves room for new investigations.

Since, geometry of conic sections and other triangle curves are broadly used in the projective geometry we looked on the obtained result through the prism of the projective geometry.

According to the Pascal's theorem if six arbitrary points are chosen on a conic and joined by line segments in any order to form a hexagon, then the three pairs of opposite sides of the hexagon meet at three points which lie on a straight line.

Let‘s consider the first derived triangle curve based on Jerabek hyperbola

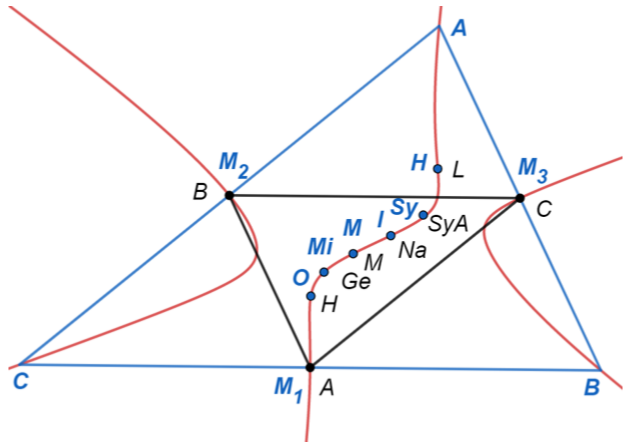

Figure 6: New cubic based on Lucas cubic for the medial triangle

Points in the Euler triangle	**Points in the base triangle**
I (incenter)	M_{IH} (midpoint of incener and orthocenter)
M (centroid)	M_{MH} (midpoint of centroid and orthocenter)
O (circumcenter)	E (nine-point center)
H (orthocenter)	H (orthocenter)
N (Nagel point)	F (Furhman point)
L (de Longchamps point)	O (circumcenter)

Table 9: Correspondence table between points in the Euler ad base triangles

for the excentral triangle. New hyperbola passes though excenters, Bevan point, incenter, mittenpunkt, de Longchaps point. We bulit a hexagon with verices in the given triangle centers and apply Pascal's theorem.

Let I_1, I_2 be excenters, and Be, Mi, L be Bevan point, Mi mittenpunkt, de Longchamps point, respectively. We get the following results:

Corollary 9. *Concurent points of I_2Be and LI, $BeMi$ and I_1L, MiI and I_2I_1 belong to one line.*

Corollary 10. *Concurent points of segments I_2Mi and BeI_1, BeL and II_2, MiL and II_2 lie on one line.*

Darboux cubic in the Euler triangle	**New cubic for the base triangle**
A, B, C (vertices)	E_1, E_2, E_3 (vertices of the Euler triangle)
A', B', C' (antipods of the triangle vertices)	M_1, M_2, M_3 (middles of the triangle sides)
I (incenter)	M_{IH} (midpoint of incenter and orthocenter)
O(circumcenter)	E (nine-point center)
H (orthocenter)	H (orthocenter)
L (de Longchamps point)	O (circumcenter)

Table 10: Mtaching points for the Darboux cubic and Euler triangle

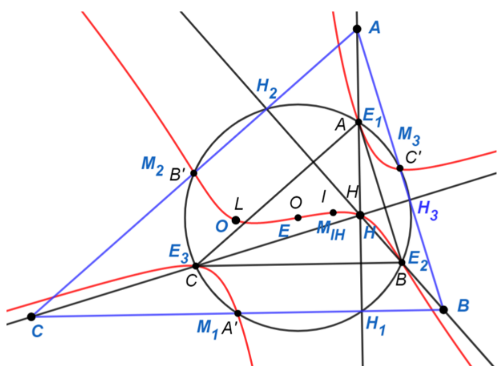

Figure 7: New cubic based on Darboux cubic in the Euler trianle

Simiraly, we have applied the same idea for the hexagon inscribed in the new hyperbola derived from the Jerabek hyperbola for the mid-arc triangle.

Let A_2, A_3 be middles of the arcs of the circumcircle, and Be, I, S, O be Bevan point, incenter, Speaker point, circumcenter, respectively. The following facts were discovered:

Corollary 11. *Points of intersection of lines A_2Be and SI, IA_3 and OA_2, BeA_3 and OS belong to one line.*

Corollary 12. *Points of intersection of line segments BeO and A_3A_2, BeS and IA_2, A_3S and IA_2 belong to a straight line.*

Points for the mid-arc triangle	**Points for the base triangle**
O (circumcenter)	O (circumcenter)
H (orthocenter)	I (incenter)
Sy (Lemoine center)	M_{MiI} (midpoint of mittenpunkt and incenter)
L (de Longchamps point)	Be (Bevan point)
K (Kosnita point)	S (Schiffler point)

Table 11: Correspondence table between points in the mid-arc and base triangles

Jerabek hyperbola for mid-arc triangle	**New hyperbola for the base triangle**
A, B, C (vertices)	A_1, A_2, A_3 (middles of the arcs of the circumcircle)
O (circumcenter)	O (circumcenter)
H (orthocenter)	I (incenter)
Sy (Lemoine point)	M_{MiI} (midpoint of mittenpunkt and inenter)
K (Kosnita point)	S (Schiffler point)
L' (isogonal conjugate of the de Longchamps point)	Be' (isogonal conjugate of the Bevan point)

Table 12: Mtaching points for the Jerabek hyperbola and Euler triangle

Moreover, combination of two of the discovered triangle cubics gives us very interesting corollary as well. Let's consider new cubic derived from the Darboux cubic for the excentral triangle and new cubic constructed with the base Darboux cubic with respect to the medial triangle. The first mentioned new cubic passes through bases of altitudes, vertices, orthocenter, nine-point center, circumcenter, let's name it $P(x, y)$. The second mentioned new cubic passes through middles of the triangle sides, Speaker point, nine-point center, circumcenter, orthocenter, let‘s name it $Q(x, y)$. We may notice that this two cubics pass through three common points which are nine-point center, circumcenter and orthocenter. Moreover, this three points belong to Euler line, let it has an equation $ax + by + c = 0$. Since, we have two cubics which

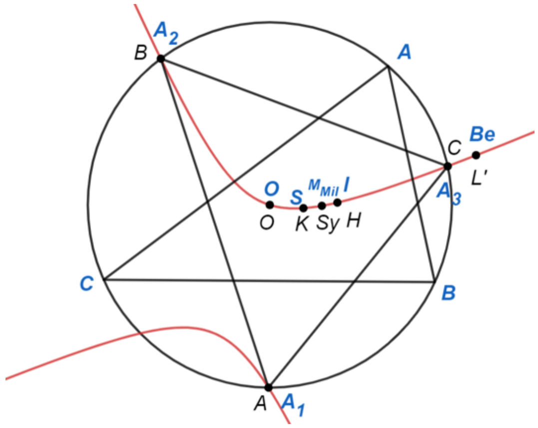

Figure 8: New cubic based on the Jerabek hyperbola for the mid-arc triangle

pass through points which belong to one line, there exists such integer t such that the following holds: $P(x, y) - tQ(x, y) \vdots ax + by + c$. Therefore, Euler line is linear component of the composition of two new cubics. In addition, points of intersection of the linear component with the curve are inflection points [11].

Corollary 13. *Euler line is a linear component of the composition of new triangle cubic (passes through bases of altitudes, vertices, orthocenter, nine-point center, circumcenter) and new triangle cubic (passes through middles of the triangle sides, Speaker point, nine-point center, circumcenter, orthocenter). Moreover, orthocenter, circumcenter, and nine-point center are inflection point of the composition of these two curves.*

The above corollaries prove that the discovered in the research new triangle curves could be applied in different geometric areas and studied in advanced.

Remark 14. A further continue of our research consists in the same analysis of singularities as provided by second author in [3; 10] for cubic obtained by us in the presented work.

3. Conclusion

During the research were discovered three new triangle conics and five new triangle cubics, what is very significant result for the classical geometry. In addition, was shown that proceedings of the study could be applied not only in Euclidian space, but in projective as well. However, the main result of the research was developed algorithm of deriving new triangle curves. This opens an opportunity for creating more triangle curves, while applying the method for various triangles, points, and geometric constructions.

The developed idea significantly simplifies the question of creating curves passing through triangular centers. However, it opens up a number of new questions. Which interesting properties do new curves have? What is the topological nature of these transformations? Is it possible to apply a similar idea to non-Euclidean objects? Could one use the same method over an arbitrary finite field? Can this idea be further generalized?

Скуратовський Р., Стародуб В.
Трикутні коніки і кубіки

Резюме

В статті описано трикутні кубики та коніки в класичній геометрії з елементами проективної геометрії. В останні роки цією темою з алгебраїчної точки зору активно займався Н. Дж. Вільдбергер. Трикутні коніки недавно були детально вивчені Х. М. Канді і К. Ф. Паррі. Основне завдання статті — розробити алгоритм побудови кривих, що проходять через центри трикутників. В ході дослідження було помічено, що кілька різних центрів трикутників в різних трикутниках збігаються. Найпростіший приклад: центр в базовому трикутнику — це ортоцентр в ексцентральному трикутнику. Це було ключем до створення алгоритму. Дійсно, ми можемо порівняти точки, що належать одній кривій (базовій кривій), з іншими точками іншого трикутника. Таким чином, ми отримуємо новий цікавий геометричний об'єкт. В ході дослідження було виведено ряд нових трикутних конік і кубік, розглянуто їх властивості в евклідовому просторі. Крім того, обговорювалися наслідки отриманих теорем з проективної геометрії, що доводить, що всі отримані результати можуть бути перенесені на проективну площину.
Ключові слова: трикутні кубики, коніки, криві, проективна геометрія, простір Евкліда.

Скуратовский Р., Стародуб В.
Треугольные коники и кубики

Резюме

В статье изучены треугольные кубики и коники в классической геометрии с элементами проективной геометрии. В последние годы этой темой с алгебраической точки зрения активно занимался Н. Дж. Вильдбергер. Треугольные коники недавно были подробно изучены Х. М. Канди и К. Ф. Парри. Основная задача статьи — разработать алгоритм построения кривых, проходящих через различные центры треугольников. В ходе исследования было замечено, что несколько разных центров треугольников в разных треугольниках совпадают. Самый простой пример: центр в базовом треугольнике — это ортоцентр в эксцентральном треугольнике. Это было ключом к созданию алгоритма. Действительно, мы можем сопоставить точки, принадлежащие одной кривой (базовой кривой), с другими точками другого треугольника. Таким образом, мы получаем новый интересный геометрический объект. В ходе исследования был выведен ряд новых треугольных коник и кубик, рассмотрены их свойства в евклидовом пространстве. Кроме того, обсуждались следствия полученных теорем проективной геометрии, что доказывает, что все полученные результаты могут быть перенесены на проекционную плоскость.
Ключевые слова: треугольные кубики, коники, кривые, проективная геометрия, пространство Эвклида.

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NEWTON'S METHOD FOR THE EIGENVALUE PROBLEM OF A SYMMETRIC MATRIX

Newton's method for calculating the eigenvalue and the corresponding eigenvector of a symmetric real matrix is considered. The nonlinear system of equations solved by Newton's method consists of an equation that determines the eigenvalue and eigenvector of the matrix and the normalization condition for the eigenvector. The method allows someone to simultaneously calculate the eigenvalue and the corresponding eigenvector. Initial approximations for the eigenvalue and the corresponding eigenvector can be found by the power method or by the reverse iteration with shift. A simple proof of the convergence of Newton's method in a neighborhood of a simple eigenvalue is proposed. It is shown that the method has a quadratic convergence rate. In terms of computational costs per iteration, Newton's method is comparable to the reverse iteration method with the Rayleigh ratio. Unlike reverse iteration, Newton's method allows to compute the eigenpair with better accuracy.
MSC: 65F15.
Key words: Newton's method, eigenvalue, symmetric matrix, reverse iteration.
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1. Introduction

If a sufficiently good approximation to the solution of the equation $F(x) = 0$ is known, then the Newton method is an effective method for increasing the accuracy of approximation. Many statements about the convergence of Newton's method come from the well-known results of L.V. Kantorovich, who transferred Newton's method to nonlinear operator equations in Banach spaces[1].

The application of Newton's method to spectral problems of matrices has a long history. Without pretending to be complete, we can note some stages. J. H. Wilkinson [2] investigated the application of Newton's method to find the roots of the characteristic equation $det(A - \lambda I) = 0$. In the monograph by D.K. Faddeev. and V.N. Faddeeva [3] Newton's method is applicable to refine an individual eigenvalue and its own eigenvector, the first component of which is not vanishingly small in comparison with the others, so that without loss of generality it can be considered equal to unity. A nonlinear equation

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is obtained for the eigenvalue. Newton's method is applied to this equation. In V. N. Kublanovskaya's article [4], an algorithm for finding the complex conjugate eigenvalues and eigenvectors of a real matrix in real arithmetic is constructed. To calculate the real and imaginary parts of the eigenvalue the roots of nonlinear equations are found by Newton's method. In all of the above cases, Newton's method was applied to scalar nonlinear equations to refine the required eigenvalue. L. Kollatz [5] described a different approach. The eigenvalue problem $Ax = \lambda Bx$ (A, B — given matrices of dimension $n \times n$) is reduced to finding the roots of a nonlinear system from the $n+1$ equation:

$$\begin{cases} Ax - \lambda Bx = 0, \\ x_n - 1 = 0. \end{cases} \tag{1}$$

To prove the convergence of Newton's method for system (1), it is proposed to use the theorem on the convergence of Newton's method for a nonlinear operator equation in a Banach space. However, in practice, it is difficult to prove the fulfillment of the conditions of this theorem. In the same place the proof of the convergence of Newton's method is given only for a numerical example with matrices of size 3×3. Moreover, the proof uses the estimates obtained on the basis of the values found in the process of calculations. The authors have not found any other proofs of the convergence of Newton's method for systems of the form (1). We have proposed a simple proof of the convergence of Newton's method for a system of the form (1), which is based on the known theorem on the convergence of Newton's method for a system of nonlinear equations [6].

2. Newton's method for the eigenvalue problem

The Newton method is introduced for the equation

$$F(x) = 0, \tag{2}$$

where $F : R^n \to R^n$ is a smooth mapping. Let $x^k \in R^n$ be the current approximation to the desired solution $\overline{x}$ of equation (2). Then the approximation x^{k+1} is found from the linear approximation of equation (2) near x^k,

$$F(x^k) + F'(x^k)(x - x^k) = 0,$$

and Newton's method is written as

$$x^{k+1} = x^k - (F'(x^k))^{-1} F(x^k), \quad k = 0, 1, \ldots. \tag{3}$$

In [6] the following theorem is proved.

Theorem 1. *Let the map $F : R^n \to R^n$ be differentiable in some neighborhood of the point $\overline{x} \in R^n$, and its derivative is continuous at this point. Let $\overline{x}$ be a solution to equation (2), and $\det F'(\overline{x}) \neq 0$. Then for any initial approximation $x^0 \in R^n$ sufficiently close to $\overline{x}$, Newton's method (3) defines a sequence converging to $\overline{x}$. The rate of convergence is superlinear, and if the derivative F is continuous in the Lipschitz sense in a neighborhood of the point $\overline{x}$, then it is quadratic.*

The eigenvalues and the corresponding eigenvectors of the symmetric matrix $A \in R^{n \times n}$ are the roots of the nonlinear system:

$$\begin{cases} Ax - \lambda x = 0, \\ \frac{1}{2}(1 - x^T x) = 0. \end{cases} \tag{4}$$

The last equation of the system is the normalization condition of the eigenvector. We write system (4) in the form (2), setting

$$F\left(\begin{bmatrix} x \\ \lambda \end{bmatrix}\right) = \begin{bmatrix} Ax - \lambda x \\ \frac{1}{2}(1 - x^T x) \end{bmatrix}. \tag{5}$$

The derivative of the mapping F is easy to calculate:

$$F'\left(\begin{bmatrix} x \\ \lambda \end{bmatrix}\right) = \begin{bmatrix} A - \lambda I & -x \\ -x^T & 0 \end{bmatrix}. \tag{6}$$

Let's define the following iterative process:

$$\begin{bmatrix} x^{k+1} \\ \lambda^{k+1} \end{bmatrix} = \begin{bmatrix} x^k \\ \lambda^k \end{bmatrix} - \left(F'\left(\begin{bmatrix} x^k \\ \lambda^k \end{bmatrix}\right)\right)^{-1} F\left(\begin{bmatrix} x^k \\ \lambda^k \end{bmatrix}\right), \quad k = 0, 1, \ldots. \tag{7}$$

Theorem 2. *Let $\overline{\lambda}$ be a simple eigenvalue, $\overline{x}$ be the corresponding eigenvector of a real symmetric matrix A. Then for any initial approximations $[x^0, \lambda^0]^T$ sufficiently close to $[\overline{x}, \overline{\lambda}]^T$, Newton's method (7) defines a sequence converging to $[\overline{x}, \overline{\lambda}]^T$ with quadratic speed.*

Proof. Let us show that for the mapping F of the iterative process (7) all conditions of Theorem 2 are satisfied. Indeed, according to (5) and (6), the mapping F is continuously differentiable, and its derivative is continuous in the Lipschitz sense in the neighborhood of $[\overline{x}, \overline{\lambda}]^T$. Let us prove that

$$\det F'\left(\begin{bmatrix} \overline{x} \\ \overline{\lambda} \end{bmatrix}\right) \neq 0.$$

It suffices to show that the system

$$\begin{bmatrix} A-\overline{\lambda}I & -\overline{x} \\ -\overline{x}^T & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8}$$

has only a trivial solution. Multiplying the first equation of system (8) on the left by x^T and taking into account the last equation, we obtain

$$x^TAx - \overline{\lambda}x^Tx = 0.$$

If we assume that $x \neq 0$, then

$$\overline{\lambda} = \frac{x^TAx}{x^Tx}$$

and $x = \alpha\overline{x}$ ($\alpha \neq 0$), because $\overline{\lambda}$ is a simple eigenvalue. But then from the last equation of system (8) $\alpha\overline{x}^T\overline{x} = 0$. It is impossible. Hence, $x = 0$. Taking this into account, the first equation of system (8) takes the form

$$-\lambda\overline{x} = 0.$$

Hence, $\lambda = 0$. □

Let λ^0, x^0 be some approximations to the required eigenpair $\overline{\lambda}$, $\overline{x}$ of a symmetric matrix A. Then, the k-th step of Newton's method (7) is conveniently written as follows:

1. find the solution $[y^k, \mu^k]^T$ of the system

$$\begin{bmatrix} A-\lambda^kI & -x^k \\ -(x^k)^T & 0 \end{bmatrix} \begin{bmatrix} y^k \\ \mu^k \end{bmatrix} = \begin{bmatrix} Ax^k - \lambda^kx^k \\ \frac{1}{2}(1-(x^k)^Tx^k) \end{bmatrix}; \tag{9}$$

2. define

$$\begin{cases} x^{k+1} = x^k - y^k, \\ \lambda^{k+1} = \lambda^k - \mu^k. \end{cases}$$

Let us note some features of Newton's method (7). The right-hand side of Eq. (9) contains the residual $r^k = (A - \lambda^k)x^k$, which is the best computational measure of the accuracy of (λ^k, x^k) as an eigenpair of the matrix A [7]. Therefore, it is convenient to define the condition for the completion of the iterative process as follows:

$$||r^k|| \leq \varepsilon, \tag{10}$$

where ε is the required computational accuracy.

In terms of computational costs, Newton's method (7) is comparable to the reverse iteration with the Rayleigh ratio, the k-th step of which has the form [7]

$$\begin{aligned} &1. \quad \rho^k = (x^k)^T A x^k, \\ &2. \quad (A - \rho^k) y^{k+1} = x^k, \\ &3. \quad x^{k+1} = y^{k+1} / ||y^{k+1}||_2. \end{aligned}$$

Indeed, at each iteration, the main computational costs of the methods are associated with solving the system, the matrix of which changes with the use of the shift λ^k or ρ^k. Newton's method may be preferable to reverse iteration with the Rayleigh ratio in the following case. If the eigenvalue is computed by reverse iteration with high precision, then the matrix $A - \rho^k I$ becomes degenerate in machine arithmetic and the calculations should be interrupted. It may happen that the corresponding eigenvector has not yet been calculated with a given precision. As proved above, the matrix of system (9) is nondegenerate, even if λ^k coincides with the desired eigenvalue. Therefore, calculations by Newton's method can be continued to achieve the required accuracy of the eigenvector even if the eigenvalue has already been calculated exactly.

3. Numerical experiments

The finite-difference approximation of the spectral problem for the Laplace operator in the unit rectangle with homogeneous Dirichlet conditions is an eigenvalue problem for the symmetric matrix A. All eigenvalues of the matrix A are different. The minimum eigenvalue and its corresponding eigenvector are defined as follows [8]:

$$\lambda_h = \frac{8}{h^2} \sin^2 \frac{\pi h}{2},$$

$$\varphi_h(x_i, y_i) = 2 \sin(\pi x_i) \sin(\pi y_i), \quad x_i = ih,\ y_j = jh,\ i, j = 1, \ldots, N-1.$$

Here the integer N defines the parameter $h = 1/N$ of the uniform mesh on the unit rectangle.

Calculations have been performed by the MATLAB package. The minimum eigenvalue and the corresponding eigenvector of the matrix A of size $10^5 \times 10^5$ ($N = 101$) have been calculated by the Newton method and inverse iteration with the Rayleigh ratio. To determine the initial approximations λ^0 and x^0, one step of reverse iteration had been performed for the initial vector $y^0 = [1, \ldots, 1]^T$. The calculation results are presented in tables 1 and 2. Let's note the following. In three steps of reverse iteration with the Rayleigh relation, the matrix $A - \rho^h I$ becomes degenerate in machine arithmetic and the calculations are terminated. In Newton's method, the condition number of the matrix of system (9) does not increase when approaching the eigenvalue and calculations can be continued to achieve better accuracy.

Table 13: Reverse iteration with the Rayleigh ratio.

k	$\|\|r^k\|\|_2$	$\lambda_h - \rho^k$	$\|\|\varphi_h - x^k\|\|_2$	$cond(A - \rho^k I)$
1	12.2	-0.901	2.00	9.04e+04
2	0.0895	-9.64e-05	1.33e-09	8.46e+08
3	1.06e-07	-1.42e-14	*	6.08e+16

Table 14: Newton's method.

k	$\|\|r^k\|\|_2$	$\lambda_h - \lambda^k$	$\|\|\varphi_h - x^k\|\|_2$	$cond\left(\begin{bmatrix} A - \lambda^k I & -x^k \\ -(x^k)^T & 0 \end{bmatrix}\right)$
1	14.2	0.0874	0.0125	6.83e+05
2	0.900	5.04e-04	7.801e-05	8.47e+04
3	0.00108	3.88e-08	3.05e-09	8.16e+04
4	3.93e-08	-3.55e-14	1.84e-15	8.16e+04
5	4.25e-12	-7.11e-15	1.77e-15	8.16e+04

4. Conclusion

Newton's method for calculating the eigenvalue and the corresponding eigenvector of a symmetric real matrix is presented in this study. The proof of

the quadratic rate convergence of Newton's method in a neighborhood of a simple eigenvalue is given. In terms of computational costs per iteration, Newton's method is comparable to the reverse iteration method with the Rayleigh ratio. The most attractive feature of this method is that it allows to compute the eigenpair with good accuracy. Proving of the applicability of the method for multiple eigenvalues and for asymmetric matrices is the prospects for further research in this direction.

Вербіцький В. В., Гук А. Г.

Метод Ньютона для задачі на власні значення симетричної матриці

Резюме

Розглянуто метод Ньютона обчислення власного значення та відповідного власного вектора дійсної симетричної матриці. Нелінійна система рівнянь, яка розв'язується методом Ньютона, складається з рівняння, що визначає власне значення і власний вектор матриці, та умови нормування власного вектора. Метод дозволяє одночасно обчислювати власне значення і відповідний власний вектор. Початкові наближення для власного значення і відповідного власного вектора можна знайти степеневим методом або методом зворотної ітерації зі зсувом. Запропоновано простий доказ збіжності методу Ньютона в околиці простого власного значення. Показано, що метод має квадратичну швидкість збіжності. За обчислювальними витратами на одну ітерацію метод Ньютона можна порівняти з методом зворотної ітерації з відношенням Релея. На відміну від зворотної ітерації, метод Ньютона дозволяє обчислити власну пару з більшою точністю.

Ключові слова: Метод Ньютона, власне значення, симетрична матриця, зворотна ітерація.

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Метод Ньютона для задачи на собственные значения симметричной матрицы

Резюме

Рассмотрен метод Ньютона вычисления собственного значения и соответствующего собственного вектора симметричной вещественной матрицы. Нелинейная система уравнений, решаемая методом Ньютона, состоит из уравнения определяющего собственное значение и собственный вектор матрицы и условия нормировки собственного вектора. Метод позволяет одновременно вычислять собственное значение и соответствующий собственный вектор. Начальные приближения для собственного значения и соответствующего собственного вектора можно найти степенным методом или обратной итерацией со сдвигом. Предложено простое доказательство сходимости метода в окрестности простого собственного значения. Показано, что метод обладает квадратичной скоростью сходимости. По вычислительным затратам на одну итерацию метод Ньютона сравним с методом обратной итерации с отношением Релея. В отличие от обратной итерации, метод Ньютона позволяет вычислять собственную пару с большей точностью.

Ключевые слова: Метод Ньютона, собственное значение, симметричная матрица, обратная итерация.

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ПОБУДОВА МНОЖИНИ ДОСЯЖНОСТІ ДИНАМІЧНОЇ СИСТЕМИ В $\mathbb{R}^3$

У статті запропоновані два алгоритми чисельної побудови опуклої оболонки множини в тривимірному просторі що використовують його опорну функцію. Проведено порівняння алгоритмів, знайдено асимптотичні оцінки. Показано застосування запропонованого апарату до знаходження множини досяжності для динамічних систем.
MSC: 00-02, 00A05, 00A06.
Ключові слова: опукла оболонка, множина досяжності, диференціальні включення, опорна функція.
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1. Вступ

Задачі оптимального керування виникають при дослідженні процесів в різноманітних сферах, таких як економіка, біологія, медицина, техніка і т.д. При аналізі систем з керуванням одним з ключових підходів є побудова множини досяжності, що дає інформацію про всі можливі стани системи в кожний момент часу. Також з задачею побудови множини досяжності ми зустрічаємось при побудові розв'язку дискретних включень. Багатозначні системи, що описуються дискретними включеннями стають все більш актуальними, особливо у сферах, де є невизначеність або недостатні вихідні дані. Наразі широко відомі алгоритми побудови множини досяжності в $\mathbb{R}^2$. В даній роботі запропоновано два чисельних алгоритми, побудови множини досяжності в $\mathbb{R}^3$. Перший алгоритм базується на апараті опорних функцій, другий за основу бере функцію деформації.

У статті будуть використовуватися такі позначення: R^n — n-вимірний евклідів векторний простір з елементами $x = (x^{(1)}, \ldots, x^{(n)})$, $\| \cdot \|$ — евклідова норма, $< \cdot, \cdot >$ — скалярний добуток, який породжує евклідову норму в $\mathbb{R}^n$, $comp(\mathbb{R}^n)$ — множина всіх непустих компактів з $\mathbb{R}^n$, $intA$ — сукупність внутрішніх точок множини $A \subset \mathbb{R}^n$, δA — границя множини $A \subset \mathbb{R}^n$, $c(F, \psi) = \sup_{x \in F} < x, \psi >$, $F \subset \mathbb{R}^n$, $\psi \in \mathbb{R}^n$ — опорна

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функція множини F, $S = \{x \in \mathbb{R}^n \colon \|x\| = 1\}$ — одинична сфера з $\mathbb{R}^n$, $S_r(a) = \{x \in \mathbb{R}^n \colon \|x - a\| \le r\}$ — замкнута куля радіуса r з центром в $a \in \mathbb{R}^n$,

$$K_{a,b,c}(x_0) = \{x \in \mathbb{R}^n \colon |x^{(1)} - x_0^{(1)}| \le a, |x^{(2)} - x_0^{(2)}| \le b, |x^{(3)} - x_0^{(3)}| \le c\}$$

— замкнутий паралелепіпед з вимірами a, b, c з центром в точці $x_0 \in \mathbb{R}^n$. $F + G = \{f + g \colon f \in F, g \in G\}$ — сума компактних множин F, G, $\lambda F = \{\lambda f \colon f \in F\}$ — добуток множини F та числа $\lambda \in \mathbb{R}$, $AF = \{Af \colon f \in F\}$ — образ множини F при лінійному перетворенні, яке задано матрицею $A \in \mathbb{R}^{n \times n}$.

2. Побудова опуклої оболонки множини в $\mathbb{R}^3$

Для побудови множини досяжності динамічної системи необхідно спочатку вирішити задачу побудови опуклої оболонки множини в $\mathbb{R}^3$. Нехай множина $F \subset \mathbb{R}^3$ і відома її опорна функція $c(F, \psi)$. Запропоновано два алгоритми побудови опуклої оболонки множини F за допомогою його опорної функції $c(F, \psi)$.

2.1. Алгоритм побудови опуклої оболонки множини, що базується на перетині опорних гіперплощин

Нехай задано деяку множину $F \subset \mathbb{R}^3$. Побудуємо систему векторів $\phi \in \mathbb{R}^3$, рівномірно взятих на одиничній сфері.

Тепер розглянемо три найближчих вектори ϕ за евклідовою метрикою $(\phi^{(1)}, \phi^{(2)}, \phi^{(3)})$ і знайдемо значення опорної функції множини F за напрямками $c(F, \phi^{(1)})$, $c(F, \phi^{(2)})$, $c(F, \phi^{(3)})$. Складемо систему лінійних алгебраїчних рівнянь (СЛАР):

$$\begin{cases} x\phi_x^{(1)} + y\phi_y^{(1)} + z\phi_z^{(1)} = c(F, \phi^{(1)}) \\ x\phi_x^{(2)} + y\phi_y^{(2)} + z\phi_z^{(2)} = c(F, \phi^{(2)}) \\ x\phi_x^{(3)} + y\phi_y^{(3)} + z\phi_z^{(3)} = c(F, \phi^{(3)}). \end{cases}$$

Розв'язком СЛАР буде точка опуклої оболонки множини. Розглянувши всі такі трійки векторів ϕ, ми зможемо знайти всі точки опуклої оболонки.

Нехай $z \in [-1, 1]$. Розіб'ємо відрізок $[-1, 1]$ з кроком Δz, де $\Delta z = \frac{2}{m}$ і m — число взятих точок на цьому інтервалі. Зафіксуємо z_1 з проміжку

$[-1, 1)$ і $z_2 = z_1 + \Delta z$. Знайдемо x_1, y_1 і x_2, y_2 такі, що $x_1^2 + y_1^2 = 1 - z_1^2$ і $x_2^2 + y_2^2 = 1 - z_2^2$. Отримаємо

$$x_1 = \sqrt{1 - z_1^2} \cos \alpha, y_1 = \sqrt{1 - z_1^2} \sin \alpha,$$

$$x_2 = \sqrt{1 - z_2^2} \cos \alpha, y_2 = \sqrt{1 - z_2^2} \sin \alpha,$$

де $\alpha \in [0, 2\pi]$. На рисунку 1 зображено множину точок границі опуклої оболонки множина F для обраних рівнів z_1 і z_2.

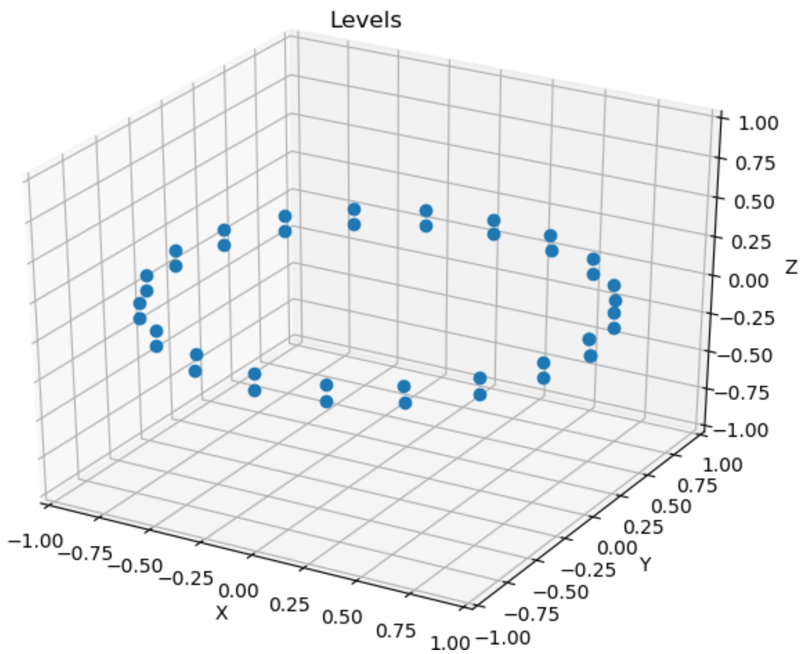

Рис. 1: Ілюстрація ітерації алгоритму

Зафіксуємо кут α і оберемо по одній точці з верхнього і нижнього рівнів, що відповідають цьому куту (на Рис. 2 це точки A і B). Будемо переміщатися по одній точці в напрямку збільшення кута на нижньому або верхньому рівні (на Рис. 2 переходимо до точки C на верхньому рівні, яка є найближчою із побудованих точок до прямої AB). Знайдемо за трикутником точку опуклої оболонки на даному рівні, яка відповідає п.с. і є найближчою до неї. Будемо продовжувати, поки не пройдемо весь рівень. Така процедура здійснюється для кожного рівня z (Рис. 2).

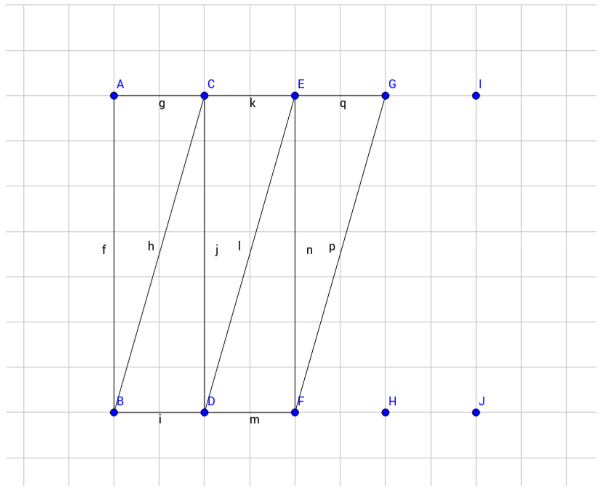

Рис. 2: Ілюстрація пошуку трикутників

За побудовою, кожна точка опуклої оболонки знайдена на i-тому рівні (рівень — два яруси точок одиничної сфери) буде з більшою компонентою z, ніж точки знайдені на рівні $i-1$. Таким чином, на кожному ярусі будуть побудовані трикутники. Знайдені трикутники додаємо в список граней триангуляції опуклої оболонки множини F (Рис. 3).

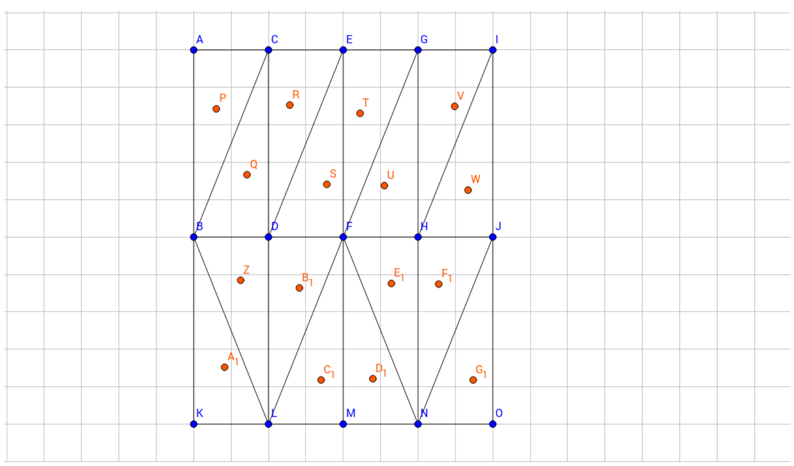

Рис. 3: Побудова триангуляції

Складність алгоритму — $O(n)$, де n — кількість векторів з S взятих при апроксимації.

Приклад 1. *Побудова опуклої оболонку множини* $F = K_{1,1,1}((0,0,0)) +$

$S_1((0,0,0))$ *за допомогою методу перетину гіперплощин зображена на Рис. 4.*

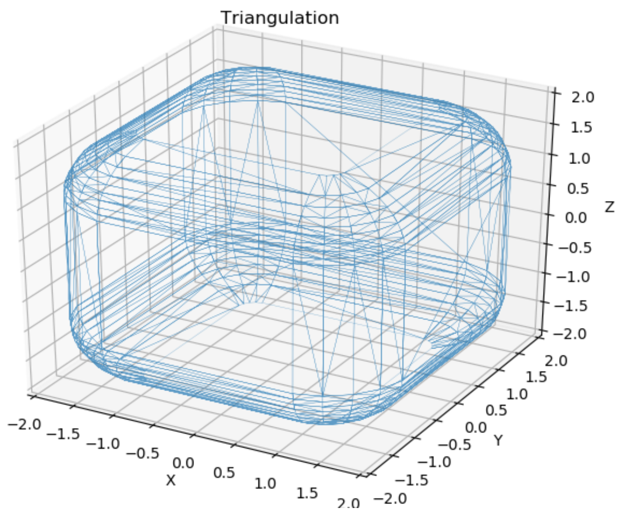

Рис. 4: Приклад побудови опуклоі множини $F = K_{1,1,1}((0,0,0)) + S_1((0,0,0))$.

Приклад 2. *Побудова опуклої оболонки множини* $F = S_1((0,0,0))$ *за допомогою методу перетину гіперплощин зображена на Рис. 5.*

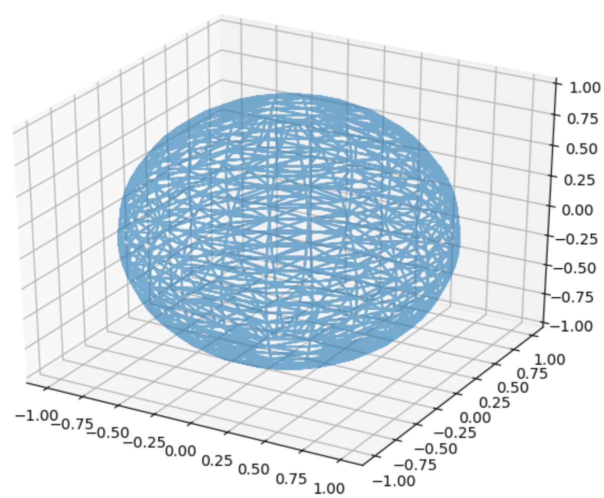

Рис. 5: Приклад побудови опуклоі множини $F = S_1((0,0,0))$.

2.2. Алгоритм побудови опуклої оболонки множини з використанням функції деформації

Означення 1. *Функцією деформації опуклої множини* $A \subset conv(\mathbb{R}^3)$, $0 \in intA$ *називається функція*

$$d(A, \phi) = sup\{\lambda > 0 : \lambda\phi \in A\}, \phi \in S.$$

Використовуючи функцію деформації, множину $A \subset conv(\mathbb{R}^3)$ можна представити у вигляді

$$A = \bigcup_{\phi \in S} \{x \in \mathbb{R}^n \colon x = \lambda\phi,\ \lambda \in [0, d(A, \phi)]\}.$$

За властивістю опорної функції вектор f належить множині X тоді і тільки тоді, коли виконується нерівність:

$$(f, \psi) \leq c(X, \psi)$$

для всіх $\psi \in S$. Тоді для кожного напрямку можна знайти максимальне λ використовуючи, що

$$(\lambda\phi, \psi) \leq c(X, \psi) \quad \forall\psi \in S\ \forall\phi \in S.$$

Таким чином, отримуємо алгоритм знаходження функції деформації. Наведемо псевдокод алгоритму:

```
λ ← inf
for all  ψ ∈ S  do
    λ_curr ← c(X,φ)/(ψ,φ)
    if  then(ψ, φ) ≠ 0  &  λ_curr > 0  &  λ > λ_curr
        λ ← λ_curr
    end if
end for
```

Складність алгоритму — $O(n^2)$, де n — кількість векторів з S взятих при апроксимації.

Після завершення роботи алгоритму можна запустити алгоритм побудови триангуляції, проте в даному випадку, для коректної візуалізації цього не потрібно.

Приклад 3. *Побудова опуклої оболонки множини*

$$F = K_{1,1,1}((0,0,0)) + S_1((0,0,0))$$

за допомогою функції деформації представлена на Рис. 6.

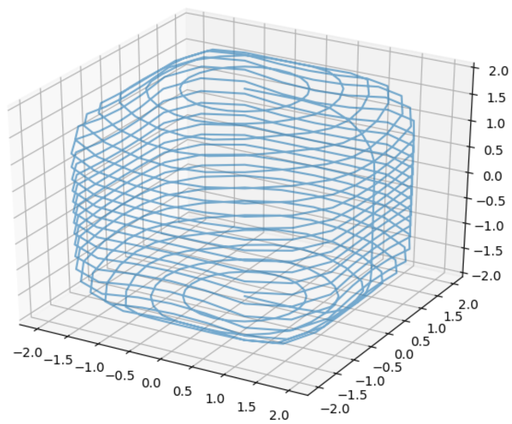

Рис. 6: Приклад побудови опуклої множини за допомогою функції деформації (без триангуляції).

Приклад 4. *Побудова опуклої оболонки множини*

$$F = S_1((0,0,0))$$

за допомогою функції деформації представлена на Рис. 7.

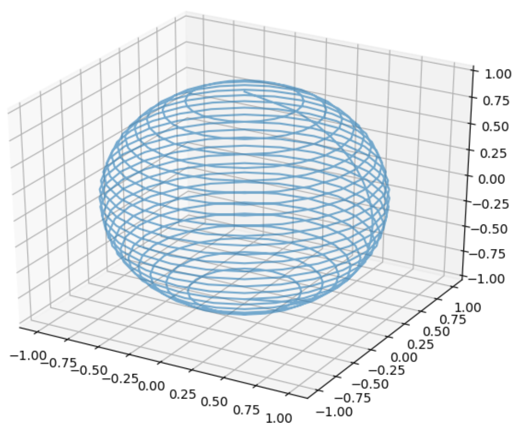

Рис. 7: Приклад побудови опуклої множини за допомогою функції деформації (без триангуляції).

2.3. Порівняння двох методів

Метод перетину гіперплощин має набагато кращу часову оцінку, обчислення всієї опуклої оболонки залежить лінійно від кількості точок, обраних при розбитті одиничної сфери. Однак, його застосування до складних фігур пов'язане з появою артефактів в триангуляції. Другий метод має квадратичну часову складність, так як на кожному кроці потрібно обчислювати коефіцієнти на підставі властивості належності точки до множини за допомогою опорної функції. Проте простота його базових обчислень дозволяє в високою точністю відтворювати складні фігури.

3. Побудова множини досяжності в $\mathbb{R}^3$

3.1. Постановка задачі

Розглянемо лінійне диференціальне включення

$$\frac{dx}{dt} \in A(t)X + F, \quad x(t_0) \in X_0, \tag{1}$$

$t \in I \subset R$ — час, $x \in \mathbb{R}^4$ — фазовий вектор, $A(t)$ — неперервна $n \times n$ матриця, $F\colon I \to conv(\mathbb{R}^n)$ — неперервне багатозначне відображення, $X_0 \in conv(\mathbb{R}^n)$.

R-розв'язком включення (1) називається абсолютно неперервне відображення $R\colon I \to conv(\mathbb{R}^n$, $R(t_0) = X_0$, якщо

$$\lim_{\sigma \downarrow 0} \frac{1}{\sigma} h \left(R(t+\sigma), \bigcup_{x \in R(t)} \{x + \sigma F(t,x)\} \right) = 0 \tag{2}$$

при майже всіх t.

Розглянемо питання побудови R-розв'язку включення (1). Оскільки $A(t)$ — неперервна $n \times n$ матриця, а $F\colon I \to conv(\mathbb{R}^n)$ — неперервне багатозначне відображення, то $A(t)x + F(t)$ — неперервне багатозначне відображення, що приймає значення в $conv(\mathbb{R}^n)$. Тоді розв'язок існує. Крім того, оскільки

$$h(A(t)x_1 + F(t), A(t)x_2 + F(t)) = h(A(t)x_1, A(t)x_2) \le \|A(t)\| \|x_1 - x_2\|,$$

то неперервне багатозначне відображення $A(t)x + F(t)$ задовольняє умову Ліпшиця по x, розв'язок єдиний та збігається з множиною досяжності в момент часу t з початковою умовою X_0. Таким чином, $R(t) = X(t, X_0)$, де $X(t, X_0)$ — множина досяжності. Тоді маємо

$$R(t) = e^{\int\limits_{t_0}^{t} A(s)ds} X_0 + \int\limits_{t_0}^{t} e^{\int\limits_{s}^{t} A(s)ds} F(s)ds$$

його опорна функція:

$$c(R(t), \psi) = c\left(X_0, \left(e^{(t-t_0)A}\right)^T \psi\right) + \int\limits_{t_0}^{t_1} c\left(F(s), \left(e^{(t-s)A}\right)^T \psi\right) ds.$$

3.2. Побудова розв'язку лінійного диференційного включення в $\mathbb{R}^3$

Скориставшись запропонованими вище алгоритмами побудови опуклої оболонки множини, можна побудувати розв'язок включення (1). Це буде послідовність тривимірних фігур.

Приклад 5. *Нехай задана динамічна система виду* (1) *з параметрами*

$$t_0 = 0, \quad t_1 = 1, \quad X(t_0) = K_{1,2,3}((10, 10, 10)), \quad F = S_3((0, 0, 0)), \quad A = 2I$$

Побудуємо R-розв'язок включення (1)*, використовуючи алгоритм перетину гіперплощин. Розв'язок включення* (1) *для моментів часу* 0*;* 0,25*;* 0,5*;* 0,75*;* 1 *представлено на Рис. 8.*

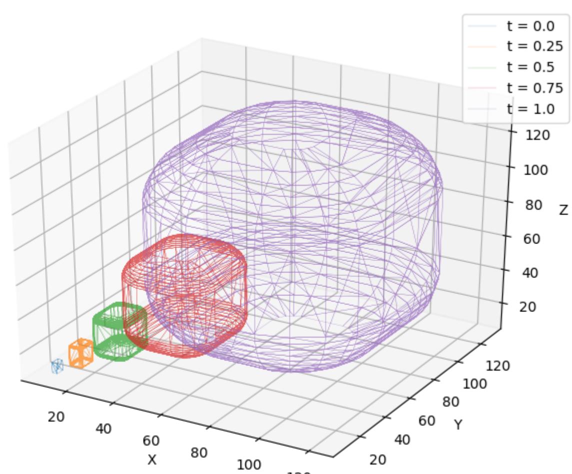

Рис. 8: Приклад побудови множини досяжності методом перетину гіперплощин.

Приклад 6. *Нехай задана динамічна система виду* (1) *з параметрами*

$$t_0 = 0, \quad t_1 = 1, \quad X(t_0) = K_{4,4,4}((0,0,0)), \quad F = S_2((0,0,0)), \quad A = I$$

Побудуємо R-розв'язок включення (1) *з використанням алгоритму на основі функції деформації. Розв'язок включення* (1) *для моментів часу* 0*;* 0,25*;* 0,5*;* 0,75*;* 1 *представлено на Рис. 9.*

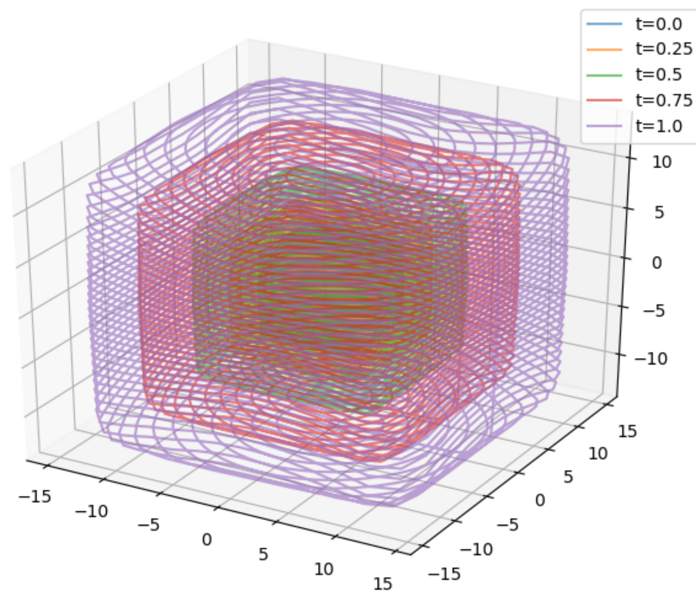

Рис. 9: Приклад побудови множини досяжності методом фукнції деформації.

4. Висновки

У статті були приведені алгоритми чисельної побудови опуклої оболонки множини в тривимірному просторі. Було проведено порівняння алгоритмів, знайдено асимптотичні оцінки, наведені приклади побудови множин в $\mathbb{R}^3$. За допомогою описаних алгоритмів була побудована множина досяжності для динамічної системи що представлена диференціальним включенням.

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Карташов Д. Г., Таирова М. С.

Построение множества достижимости динамической системы в $\mathbb{R}^3$

Резюме

В статье предложены два алгоритма численного построения выпуклой оболочки множества в трехмерном пространстве используя его опорую функцию. Проведено сравнение алгоритмов, найдены асимптотические оценки. Показано применение предложенного аппарата к нахождению множества достижимости для динамических систем.
Ключевые слова: выпуклая оболочка, множество достижимости, дифференциальные включения, опорная функция.

Kartashov D. G., Tairova M. S.

Construction of the destination set of a dynamic system in $\mathbb{R}^3$

Summary

The article proposes two algorithms for the numerical construction of the convex hull of a set in three-dimensional space using its support function. The first uses the hyperplane intersection method to find the pivot points of a set. The second one is based on the deformation function and allows you to find an arbitrary point of the convex hull of a set, which is convenient in many applications. The algorithms are compared, and asymptotic complexities are found. The application of the proposed apparatus to finding the destination set of dynamical systems is shown. The dynamic system will be based on differential inclusion.
Key words: convex hull, destination set, differential inclusions, support function.

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Дослідження в математиці і механіці. — 2020. — Т. 25, вип. 2(36). — С. 95–102

УДК 517.983

С. А. Щёголев, В. В. Карапетров

ОБ ОДНОМ КЛАССЕ РЕШЕНИЙ КВАЗИЛИНЕЙНЫХ МАТРИЧНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

Для квазилинейного матричного дифференциального уравнения, коэффициенты которого представимы в виде абсолютно и равномерно сходящихся рядов Фурье с медленно меняющимися коэффициентами и частотой, получены достаточные условия существования решения аналогичной структуры

MSC: 15A24, 34C15.

Ключевые слова: матрица, дифференциальное уравнение, квазилинейный.

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1. Основные обозначения и определения

Матричные дифференциальные уравнения издавна привлекали внимание математиков, этим уравнениям посвящено множество работ. Из вышедших в последнее время отметим [1–5]. В настоящей статье построен аналог результатов работы [6] для квазилинейных матричных дифференциальных уравнений.

Пусть

$$G(\varepsilon_0) = \{t, \varepsilon : \ t \in \mathbb{R}, \ \varepsilon \in (0, \varepsilon_0), \ \varepsilon_0 \in \mathbb{R}^+\}$$

Определение 1. Скажем, что функция $f(t, \varepsilon)$ принадлежит классу $S(m; \varepsilon_0)$, $m \in \mathbb{N} \cup \{0\}$, если:

1) $f : G(\varepsilon_0) \to \mathbb{C}$,

2) $f(t, \varepsilon) \in C^m(G(\varepsilon_0))$ по t,

3) $d^k f(t, \varepsilon)/dt^k = \varepsilon^k f_k(t, \varepsilon)$ $(0 \le k \le m)$,

$$\|f\|_{S(m;\varepsilon_0)} \overset{def}{=} \sum_{k=0}^{m} \sup_{G(\varepsilon_0)} |f_k(t, \varepsilon)| < +\infty.$$

Определение 2. Скажем, что функция $f(t, \varepsilon, \theta(t, \varepsilon))$ принадлежит классу $F(m; \varepsilon_0; \theta)$ $(m \in \mathbb{N} \cup \{0\})$, если

$$f(t, \varepsilon, \theta(t, \varepsilon)) = \sum_{n=-\infty}^{\infty} f_n(t, \varepsilon) \exp\left(in\theta(t, \varepsilon)\right),$$

Получена 04.01.2020

причём

1) $f_n(t,\varepsilon) \in S(m,\varepsilon_0)$ $(n \in \mathbb{Z})$,

2)

$$\|f\|_{F(m;\varepsilon_0;\theta)} \overset{def}{=} \sum_{n=-\infty}^{\infty} \|f_n\|_{S(m;\varepsilon_0)} < +\infty,$$

3) $\theta(t,\varepsilon) = \int\limits_0^t \varphi(\tau,\varepsilon)d\tau$, $\varphi \in \mathbb{R}^+$, $\varphi \in S(m,\varepsilon_0)$, $\inf\limits_{G(\varepsilon_0)} \varphi(t,\varepsilon) = \varphi_0 > 0$.

Определение 3. Скажем, что матрица $A(t,\varepsilon) = (a_{jk}(t,\varepsilon))_{j,k=\overline{1,N}}$ принадлежит классу $S_2(m;\varepsilon_0)$, $(m \in \mathbb{N} \cup \{0\})$, если $a_{jk} \in S(m;\varepsilon_0)$ $(j,k = \overline{1,N})$.

Определим норму

$$\|A(t,\varepsilon)\|_{S_2(m;\varepsilon_0)} = \max_{1\le j\le N} \sum_{k=1}^{N} \|a_{jk}(t,\varepsilon)\|_{S(m;\varepsilon_0)}.$$

Определение 4. Скажем, что матрица $B(t,\varepsilon,\theta) = (b_{jk}(t,\varepsilon,\theta))_{j,k=\overline{1,N}}$ принадлежит классу $F_2(m;\varepsilon_0;\theta)$ $(m \in \mathbb{N} \cup \{0\})$, if $b_{jk}(t,\varepsilon,\theta) \in F(m;\varepsilon_0;\theta)$ $(j,k = \overline{1,N})$.

Определим норму

$$\|B(t,\varepsilon,\theta)\|_{F_2(m;\varepsilon_0;\theta)} = \max_{1\le j\le N} \sum_{k=1}^{N} \|b_{jk}(t,\varepsilon,\theta)\|_{F(m;\varepsilon_0;\theta)}. \tag{1}$$

2. Постановка задачи

Рассмотрим квазилинейное матричное дифференциальное уравнение:

$$\frac{dX}{dt} = A(t,\varepsilon)X - XB(t,\varepsilon) + F(t,\varepsilon,\theta) + \mu\Phi(t,\varepsilon,\theta,X), \tag{2}$$

где $A(t,\varepsilon), B(t,\varepsilon) \in S_2(m;\varepsilon_0)$, $F(t,\varepsilon,\theta) \in F_2(m;\varepsilon_0;\theta)$, матрица X принадлежит некоторой замкнутой ограниченной области $D \subset \mathbb{C}^{N\times N}$, где $\mathbb{C}^{N\times N}$ – пространство комплекснозначных $(N\times N)$-матриц вещественного аргумента. Матрица-функция $\Phi(t,\varepsilon,\theta,X)$ предполагается принадлежащей классу $F_2(m;\varepsilon_0;\theta)$ относительно t,ε,θ и непрерывной по X в D. $\mu \in [0,\mu_0]$ — малый вещественный параметр.

Изучается вопрос о существовании частных решений классов $F_2(m_1;\varepsilon_1;\theta)$ $(m_1 \le m, \varepsilon_1 \le \varepsilon_0)$ уравнения (2).

3. Вспомогательные результаты.

Лемма 1. *Пусть задано скалярное линейное дифференциальное уравнение 1-го порядка*

$$\frac{dx}{dt} = \lambda(t,\varepsilon)x + u(t,\varepsilon,\theta(t,\varepsilon)) \tag{3}$$

где $\lambda(t,\varepsilon) \in S(m;\varepsilon)$, $\inf\limits_{G(\varepsilon_0)} |\mathrm{Re}\ \lambda(t,\varepsilon)| = \gamma > 0$, $u(t,\varepsilon,\theta) \in F(m;\varepsilon_0;\theta)$. *Тогда уравнение (3) имеет единственное частное решение* $x(t,\varepsilon,\theta) \in F(m;\varepsilon_0;\theta)$. *Это решение даётся формуло:*

$$x(t,\varepsilon,\theta(t,\varepsilon)) = \int\limits_T^t u(\tau,\varepsilon,\theta(\tau,\varepsilon)) \exp\left(\int\limits_\tau^t \lambda(s,\varepsilon)ds\right) d\tau,$$

где

$$T = \begin{cases} -\infty, \ if \ \mathrm{Re}\ \lambda(t,\varepsilon) \le -\gamma < 0, \\ +\infty, \ if \ \mathrm{Re}\ \lambda(t,\varepsilon) \ge \gamma > 0, \end{cases}$$

и, кроме того, существует $K_0 \in (0,+\infty)$,*такое, что:*

$$\|x(t,\varepsilon,\theta)\|_{F(m;\varepsilon_0;\theta)} \le K_0 \|u(t,\varepsilon,\theta)\|_{F(m;\varepsilon_0;\theta)}.$$

Доказательство леммы приведено в работе [7].

Лемма 2. *Пусть уравнение (2) таково, что существуют матрицы* $L_1(t,\varepsilon), L_2(t,\varepsilon) \in S_2(m;\varepsilon_0)$ *такие, что*

а) $|\det L_k(t,\varepsilon)| \ge a_0 > 0$, $(k=1,2)$,

б) $L_1^{-1}(t,\varepsilon)A(t,\varepsilon)L_1(t,\varepsilon) = D_1(t,\varepsilon) = (d_{jk}^1(t,\varepsilon))_{j,k=\overline{1,N}}$,
$L_2(t,\varepsilon)B(t,\varepsilon)L_2^{-1}(t,\varepsilon) = D_2(t,\varepsilon) = (d_{jk}^2(t,\varepsilon))_{j,k=\overline{1,N}}$,

где $D_1(t,\varepsilon), D_2(t,\varepsilon)$ – *нижние треугольные матрицы* N*-го порядка, принадлежащие классу* $S_2(m;\varepsilon_0)$.

Тогда подстановкой

$$X = L_1(t,\varepsilon)YL_2(t,\varepsilon) \tag{4}$$

уравнение (2) приводится к виду:

$$\frac{dY}{dt} = D_1(t,\varepsilon)Y - YD_2(t,\varepsilon) - \varepsilon H_1(t,\varepsilon)Y - \varepsilon YH_2(t,\varepsilon) +$$
$$+ F_1(t,\varepsilon,\theta) + \mu\Phi_1(t,\varepsilon,\theta,Y), \tag{5}$$

где

$$H_1(t,\varepsilon)=\frac{1}{\varepsilon}\ L_1^{-1}(t,\varepsilon)\frac{dL_1(t,\varepsilon)}{dt},\quad H_2(t,\varepsilon)=\frac{1}{\varepsilon}\ \frac{dL_2(t,\varepsilon)}{dt}\ L_2^{-1}(t,\varepsilon),$$

$$F_1(t,\varepsilon,\theta)=L_1^{-1}(t,\varepsilon)F(t,\varepsilon,\theta)L_2^{-1}(t,\varepsilon),$$

$$\Phi_1(t,\varepsilon,\theta,Y)=L_1^{-1}(t,\varepsilon)\Phi(t,\varepsilon,\theta,L_1(t,\varepsilon)YL_2(t,\varepsilon))L_2^{-1}(t,\varepsilon).$$

Доказательство. Чтобы убедиться в справедливости леммы, достаточно в уравнении (2) произвести подстановку (4) и использовать условия леммы.

Лемма 3. *Пусть линейное матричное уравнение*

$$\frac{dY_0}{dt}=D_1(t,\varepsilon)Y_0-Y_0D_2(t,\varepsilon)+F_1(t,\varepsilon,\theta), \tag{6}$$

где матрицы $D_1(t,\varepsilon),D_2(t,\varepsilon),F_1(t,\varepsilon,\theta)$ *те же, что и в уравнении (5), таково, что*

$$\inf_{G(\varepsilon_0)}\left|\operatorname{Re}\left(d_{jj}^1(t,\varepsilon)-d_{kk}^2(t,\varepsilon)\right)\right|\geq b_0>0\ (j,k=\overline{1,N}). \tag{7}$$

Тогда уравнение (6) имеет единственное частное решение $Y_0(t,\varepsilon,\theta)\in F_2(m;\varepsilon_0;\theta)$, *и существует* $K_1\in(0,+\infty)$ *такое, что*

$$\|Y_0(t,\varepsilon,\theta)\|_{F_2(m;\varepsilon_0;\theta)}\leq K_1\|F_1(t,\varepsilon,\theta)\|_{F_2(m;\varepsilon_0;\theta)}. \tag{8}$$

Доказательство. Пусть

$$Y_0=(y_{jk}^0(t))_{j,k=\overline{1,N}},\quad F_1(t,\varepsilon,\theta)=\left(f_{jk}^1(t,\varepsilon,\theta)\right)_{j,k=\overline{1,N}}.$$

Тогда, расписывая уравнение (6) в покомпонентной форме, придём к скалярной линейной системе дифференциальных уравнений вида:

$$\frac{dy_{jk}^0}{dt}=\sum_{s=1}^{j}d_{js}^1(t,\varepsilon)y_{sk}^0-\sum_{s=k}^{N}d_{sk}^2(t,\varepsilon)y_{js}^0+f_{jk}^1(t,\varepsilon\theta),\ j,k=\overline{1,N}.$$

Или

$$\frac{dy_{1N}^0}{dt}=\left(d_{11}^1(t,\varepsilon)-d_{NN}^2(t,\varepsilon)\right)y_{1N}^0+f_{1N}^1(t,\varepsilon,\theta),$$

$$\dots$$

$$\frac{dy_{11}^0}{dt}=\left(d_{11}^1(t,\varepsilon)-d_{11}^2(t,\varepsilon)\right)y_{11}^0-\sum_{s=2}^{N}d_{s1}^2(t,\varepsilon)y_{1s}^0+f_{11}^1(t,\varepsilon,\theta),$$

$$\frac{dy_{2N}^0}{dt} = \left(d_{22}^1(t,\varepsilon) - d_{NN}^2(t,\varepsilon)\right) y_{2N}^0 + d_{21}^1(t,\varepsilon) y_{1N}^0 + f_{2N}^1(t,\varepsilon,\theta),$$

$$\ldots$$

$$\frac{dy_{21}^0}{dt} = \left(d_{22}^1(t,\varepsilon) - d_{11}^2(t,\varepsilon)\right) y_{21}^0 + d_{21}^1(t,\varepsilon) y_{11}^0 - \sum_{s=2}^{N} d_{s1}^2(t,\varepsilon) y_{2s}^0 + f_{21}^1(t,\varepsilon,\theta),$$

$$\ldots$$

$$\frac{dy_{NN}^0}{dt} = \left(d_{NN}^1(t,\varepsilon) - d_{NN}^2(t,\varepsilon)\right) y_{NN}^0 + \sum_{s=1}^{N-1} d_{Ns}^1(t,\varepsilon) y_{sN}^0 + f_{NN}^1(t,\varepsilon,\theta),$$

$$\ldots$$

$$\frac{dy_{N1}^0}{dt} = \left(d_{NN}^1(t,\varepsilon) - d_{11}^2(t,\varepsilon)\right) y_{N1}^0 + \sum_{s=1}^{N-1} d_{Ns}^1(t,\varepsilon) y_{s1}^0 -$$

$$- \sum_{s=2}^{N} d_{s1}^2(t,\varepsilon) y_{Ns}^0 + f_{N1}^1(t,\varepsilon,\theta).$$

На основании леммы 1 с использованием условий настоящей леммы убеждаемся, что каждое из уравнений этой системы имеет решение класса $F(m;\varepsilon_0;\theta)$. И, следовательно, уравнение (6) имеет единственное решение класса $F_2(m;\varepsilon_0;\theta)$, и справедлива оценка (8).

4. Основные результаты.

Определим область:

$$\Omega = \left\{ Y \in F_2(m;\varepsilon_0;\theta) : \|Y - Y_0\|_{F_2(m;\varepsilon_0;\theta)} \le \beta;\ \beta > 0 \right\}.$$

Теорема 1. *Пусть уравнение (5) таково, что*

1) $\inf\limits_{G(\varepsilon_0)} \left| \mathrm{Re} \left(d_{jj}^1(t,\varepsilon) - d_{kk}^2(t,\varepsilon) \right) \right| \ge b_0 > 0 \ (j,k = \overline{1,N})$;

2) *матрица-функция* $\Phi_1(t,\varepsilon,\theta,Y)$ *непрерывна по* Y, *и если* $Y \in F_2(m;\varepsilon_0;\theta)$, *то* $\Phi_1(t,\varepsilon,\theta,Y)$ *также принадлежит классу* $F_2(m;\varepsilon_0;\theta)$;

3) *существует* $L(\beta) \in (0,+\infty)$ *такое, что* $\forall\ Y_1, Y_2 \in \Omega$ *выполнено неравенство:*

$$\|\Phi_1(t,\varepsilon,\theta,Y_1) - \Phi_1(t,\varepsilon,\theta,Y_2)\|_{F_2(m;\varepsilon_0;\theta)} \le L(\beta) \|Y_1 - Y_2\|_{F_2(m;\varepsilon_0;\theta)}.$$

Тогда можно указать такое $\varepsilon_1 \in (0,\varepsilon_0)$ *и такое* $\mu_1 \in [0,\mu_0)$, *что* $\forall \varepsilon \in (0,\varepsilon_1)$, *и* $\forall \mu \in [0,\mu_1)$ *уравнение (5) имеет единственное частное решение* $Y(t,\varepsilon,\theta,\mu) \in F_2(m-1;\varepsilon_1;\theta)$.

Доказательство. Решение класса $F_2(m-1;\varepsilon_1;\theta)$ уравнения (5) будем искать методом последовательных приближений, выбрав в качестве начального приближения $Y_0(t,\varepsilon,\theta)$, а последующие приближения определив как решения класса $F_2(m-1;\varepsilon_0;\theta)$ линейных неоднородных матричных уравнений:

$$\frac{dY_{k+1}}{dt} = D_1(t,\varepsilon)Y_{k+1} - Y_{k+1}D_2(t,\varepsilon) - \varepsilon H_1(t,\varepsilon)Y_k - \varepsilon Y_k H_2(t,\varepsilon) + \\ + F_1(t,\varepsilon,\theta) + \mu\Phi_1(t,\varepsilon,\theta,Y_k), \ k = 0,1,2,\ldots \quad . \tag{9}$$

Применяя обычную методику принципа сжимающих отображений [8], несложно показать, что при достаточно малом ε и достаточно малом μ все приближения (9) остаются внутри области Ω, и процесс (9) сходится по норме $\|\cdot\|_{F_2(m-1;\varepsilon_0;\theta)}$ к решению $Y(t,\varepsilon,\theta,\mu)$ класса $F_2(m-1;\varepsilon_1;\theta)$ уравнения (5).

Теорема доказана.

Непосредственным следствием теоремы 1 является теорема 2.

Теорема 2. *Пусть уравнение (2) таково, что:*

1) выполнены условия леммы 2;

2) для уравнения (5), получающегося из уравнения (2) с помощью подстановки (4), справедлива теорема 1.

Тогда существуют такие $\varepsilon_1 \in (0,\varepsilon_0)$, $\mu_1 \in [0,\mu_0)$, *что* $\forall\ \varepsilon \in (0,\varepsilon_1)$, $\forall\ \mu \in [0,\mu_1)$ *уравнение (2) имеет единственное частное решение класса* $F_2(m-1;\varepsilon_1;\theta)$.

5. Заключение

Таким образом, для квазилинейного матричного дифференциального уравнения с коэффициентами, представимыми абсолютно и равномерно сходящимися рядами Фурье с медленно меняющимися коэффициентами и частотой, получены достаточные условия существования частного решения аналогичной структуры.

Щоголев С. А., Карапетров В. В.

Про один клас розв'язків квазілінійних матричних диференціальних рівнянь

Резюме

Для квазілінійного матричного диференціального рівняння, коефіцієнти якого зображувані у вигляді абсолютно та рівномірно збіжних рядів Фур'є з повільно змінними

коефіцієнтами та частотою, отримано достатні умови існування розв'язку аналогічної структури.
Ключові слова: матриця, диференціальне рівняння, квазілінійний.

Shchogolev S. A., Karapetrov V. V.

On one class of solutions of the quasilinear matrix differential equations

Summary

In the mathematical description of various phenomena and processes that arise in mathematical physics, electrical engineering, economics, one has to deal with matrix differential equations. Therefore, these equations are relevant both for mathematicians and for specialists in other areas of natural science. Many studies are devoted to them, in which the solvability of matrix equations in various function spaces, boundary value problems for matrix differential equations, and other problems were investigated. In this article, a quasilinear matrix equation is considered, the coefficients of which can be represented in the form of absolutely and uniformly converging Fourier series with coefficients and frequency slowly varying in a certain sense. The problem is posed of obtaining sufficient conditions for the existence of particular solutions of a similar structure for the equation under consideration. For this purpose, the corresponding linear equation is considered first. It is written down in component-wise form, and, based on the assumptions made, the existence of the only particular solution of the specified structure is proved. Then, using the method of successive approximations and the principle of contracting mappings, the existence of a unique particular solution of the indicated structure for the original quasilinear equation are proved.
Key words: matrix, differential equation, quasilinear.

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