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DIVISORS OF THE GAUSSIAN INTEGERS IN NORM GROUP E_N^+

The divisor function in norm group E_n^+ is investigated, where the set E_n^+ is a multiplicative subgroup in the multiplicative group of classes of residues modulo p^n over $\mathbb{Z}[i]$. The asymptotic formula is obtained.
MSC: 11N37.
Key words: divisor function, Gaussian integers, asymptotic formula.

Introduction. Let A, B be two infinite sets of positive numbers. We define generalized function of divisors

$$\tau_{\mathrm{A,B}}(n) = \#\{(a,b) \in \mathrm{A} \times \mathrm{B} \,|ab = n\}, (n \in \mathbb{N}).$$

Usually study a behavior in average the function $\tau_{\mathrm{A,B}}(n)$, i. e. construct an asymptotic formula for $\sum_{n\leqslant x} \tau_{\mathrm{A,B}}(n)$. In the case A = B = N we have the classical Dirichlet problem of divisors. In works of Smith and Subbarao [5], Nowak [3], Varbanec and Zarzycki [6] was investigated the case A = N, $\mathrm{B} = \mathrm{B}(b_0,q) := \{b \in \mathbb{N} \,|b \equiv b_0 \ (mod\ q)\}$. In the sequel came to be consider other sets A and B.

Varbanec and Zarzycki [7], Varbanec [8], Nowak [4] generalized this problem on the case of sets A, B, which define as the sets of all positive integers each of which is norm of integer ideal in finite extension of field $\mathbb{Q}$.

In the present paper we will consider a generalized function of divisors over the ring of Gaussian integers determined in this way:

for every $w \in \mathbb{Z}[i]$ we put

$$\tau\left(w; E_n^+\right) = \sum_{\substack{\delta\,|w \\ \delta \in E_n^+}} 1,$$

where $E_n^+ := \{\alpha \in \mathbb{Z}[i] \,|\mathrm{N}(\alpha) \equiv 1\,(mod p^n)\}$, $(p \equiv 3\,(mod 4)$, p is prime).

The set E_n^+ is a multiplicative subgroup in the multiplicative group of classes of residues modulo p^nover $\mathbb{Z}[i]$.

Auxiliary arguments. Throughout the paper, α,β and γ(also with a subscript) denote Gaussian integers; $\mathrm{N}(\alpha)$, $Sp(\alpha)$ are a norm (respectively, a trace) of α, i. e. $\mathrm{N}(\alpha) = |\alpha|^2$, $Sp(\alpha) = 2Re(\alpha)$.

We denote by $G = \mathbb{Z}[i] = \left\{a + bi \in \mathbb{C} \,\middle|a,b \in \mathbb{Z}, i^2 = -1\right\}$ and G_{p^n} (respectively,$G^*_{p^n}$) an additive group of residue classes (respectively, a multiplicative group of reduced residue classes) modulo p^n over G.

Let δ_0, δ be the Gaussian rationales (i. e. δ_0, $\delta \in \mathbb{Q}[i]$) not necessarily integers and let m be a rational integer (i. e. $m \in \mathbb{Z}$). For $Res > 1$ we consider the following series

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$$\zeta_m\left(s;\delta_0,\delta\right)=\sum_{\substack{w\in G\\ w\neq-\delta_0}}\frac{e^{4mi\arg w}}{N\left(w+\delta_0\right)^s}e^{\pi iSp(\delta w)}\tag{1}$$

The function $\zeta_m\left(s;\delta_0,\delta\right)$ accepts an analytic extending on all complex plane and calls the Hecke zeta-function.

Lemma 1. The Hecke zeta-function $\zeta_m\left(s;\delta_0,\delta\right)$ has the functional equation

$$\pi^{-s}\Gamma\left(2\left|m\right|+s\right)\zeta_m\left(s;\delta_0,\delta\right)=\pi^{-(1-s)}\Gamma\left(2\left|m\right|+1-s\right)\zeta_m\left(1-s;-\delta,\delta_0\right)e^{-\pi iSp\left(\delta_0\overline{\delta}\right)}$$

(here $\overline{\delta}$ is a complexly-conjugate with δ).

Moreover, $\zeta_m\left(s;\delta_0,\delta\right)$ is an entire function if $m\neq0$ or $m=0$ and δ is not a Gaussian integer. For $m=0$ and $\delta\in G$, $\zeta_m\left(s;\delta_0,\delta\right)$ is a holomorphic function except at $s=1$, where it has a simple pole with a residue π.

For the proof in case $\zeta_m\left(s;0,0\right)$ see [1]. The proof in other cases is similar.

Corollary. $\zeta_0\left(0;\delta_0,\delta\right)=0$ if δ_0 is not a Gaussian integer.

Lemma 2. Let $\alpha\in E_n^+$. Then for every ε, $0<\varepsilon<\frac{1}{2}$ any $T>1$ in the rectangle

$$R=\{-\varepsilon\leqslant Res\leqslant1+\varepsilon,\left|Ims\right|\leqslant T\}$$

we have

$$\begin{aligned}&(s-1)^2\left[\zeta_m\left(s;0,0\right)\left(\zeta_m\left(s;\tfrac{\alpha}{p^n},0\right)-\textstyle\sum_{\beta\in\mathrm{B}}\frac{e^{4mi\arg\left(\frac{\alpha}{p^n}+\beta\right)}}{\left(N\left(\frac{\alpha}{p^n}+\beta\right)\right)^s}\right)\right]=\\&=\mathrm{O}\left(\varepsilon^{-2}\left(t^2+1\right)\left(t^2+m^2+10\right)^\theta\right)\end{aligned}\tag{2}$$

where $\theta=\frac{(1+\varepsilon)(1+\varepsilon-\sigma)}{1+2\varepsilon}$, $\sigma=Res$.

From now on B denote the set $\{0,\ \pm1,\ \pm i\}$.

A constant in symbol "O" is absolute.

Proof. This assertion follows at once from estimates for

$$(s-1)^2\left[\zeta_m\left(s;0,0\right)\left(\zeta_m\left(s;\frac{\alpha}{p^n},0\right)-\sum_{\beta\in\mathrm{B}}e^{4mi\arg\left(\frac{\alpha}{p^n}+\beta\right)}\left(N\left(\frac{\alpha}{p^n}+\beta\right)\right)^s\right)\right]$$

on vertical legs of the rectangle R and the Phragmen–Lindelof theorem.

■

Lemma 3. Let $\delta\in\mathbb{Q}\left(i\right),N\left(\delta_0\right)<1$. Then $\zeta_0\left(s;\ \delta_0,0\right)$ has the following in the Laurent expansion

$$\zeta_0\left(s;\delta_0,0\right)=\frac{\pi}{s-1}+a_0^{'}\left(\delta_0\right)+a_1\left(\delta_0\right)\left(s-1\right)+...,$$

where

$$a_0(\delta_0)=\begin{cases}\pi\gamma+4L'\left(1,\chi_4\right)\ \ if\ \delta_0\in G,\\ \pi\gamma+4L'\left(1,\chi_0\right)+\mathrm{O}\left(\min\limits_{\beta\in\mathrm{B}}\left(N\left(\delta_0+\beta\right)\right)^{-1}\right)\ \ if\ \delta_0\in\mathbb{Q}\left(i\right),\ \delta_0\notin G;\end{cases}$$

γ is the Euler's constant, χ_4 is non-principal character modulo 4.

Proof. For $\delta_0 = 0$ we have $\zeta_0(s;\, 0{,}0) = 4\xi(s)L(s{,}\chi_4)$,

where $\xi(s)$ is the Riemann zeta-function, $L(s,\chi_4)$ is the Dirichlet zeta-function with non-principal character modulo 4.

Hence,

$$a_0\left(\delta_0\right) = \pi\gamma + 4L'\left(1{,}\chi_4\right).$$

Since a residue of $\zeta_0\left(s;\delta_0{,}0\right)$ does not depends at δ_0, $\delta_0 \neq 0$, we may write

$$\begin{aligned}
&a_0\left(\delta_0\right) - a\left(0\right) = \lim\left(\zeta_0\left(s;\delta_0{,}0\right) - \zeta_0\left(s;0{,}0\right)\right) = \\
&= \lim_{s\to 1\pm 0}\left\{\tfrac{1}{\left(N(\delta_0)\right)^s} + \sum\nolimits_{N(\beta)=1}\left(\tfrac{1}{\left(N(\delta_0+\beta)\right)^s} - \tfrac{1}{\left(N(\beta_0)\right)^s}\right) + \right.\\
&\left. + \sum\nolimits_{N(\beta)\geqslant 2}\left(\tfrac{1}{\left(N(\delta_0+\beta)\right)^s} - \tfrac{1}{\left(N(\beta)\right)^s}\right)\right\} = \\
&= \tfrac{1}{N(\delta_0)} + \sum\nolimits_{N(\beta)=1}\tfrac{1}{N(\delta_0+\beta)} - 4 + \sum\nolimits_{N(\beta)\geqslant 2}\tfrac{N(\beta)-N(\beta+\delta_0)}{N(\beta)N(\beta+\delta_0)}
\end{aligned}$$

At last, if we observe that for $N\left(\beta\right)\to\infty$

$$\left|\frac{N\left(\beta\right) - N\left(\beta+\delta_0\right)}{N\left(\beta\right)N\left(\beta+\delta_0\right)}\right| \leqslant \frac{c\left|\delta_0\right|\cdot\left|\beta\right|}{N\left(\beta\right)N\left(\beta+\delta_0\right)} = \mathrm{O}\left(N\left(\delta\right)^{1/2} N\left(\beta\right)^{-3/2}\right)$$

we obtain the assertion of Lemma.

■

Corollary.

$$\underset{s=1}{res}\left\{\zeta_0\left(s;0{,}0\right)\zeta_0\left(s;\frac{\alpha}{p^n}{,}0\right)\frac{y^s}{s}\right\} = \pi^2 y\log y + c_0\left(\alpha{,}p^n\right)y \tag{3}$$

where

$$\begin{aligned}
&c_0\left(\alpha{,}p^n\right) = \pi\left(\sum\nolimits_{\beta\in \mathrm{B}}\tfrac{1}{N\left(\frac{\alpha}{p^n}+\beta\right)} + 2\pi\left(\gamma - \tfrac{1}{2}\right) + 8L'\left(1{,}\chi_4\right) - 4 + \right.\\
&\left. + \mathrm{O}\left(N^{\frac{1}{2}}\left(\tfrac{\alpha}{p^n}\right)\right)\right)
\end{aligned} \tag{4}$$

Lemma 4. For every $\varepsilon > 0$ and $T\to\infty$ the following assertion has place

$$\begin{aligned}
&\int_{-T}^{T}\left|\tfrac{e^{4mi\arg\gamma}}{N(\gamma)^s}\zeta_m\left(s;\tfrac{\alpha}{\gamma}{,}0\right) - \sum\nolimits_{\mathrm{B}} e^{4mi\arg(\alpha+\beta\gamma)}\left(N\left(\alpha+\beta\gamma\right)\right)^{-s}\right|ds = \\
&= \mathrm{O}\left(\tfrac{\left(T^2+m^2\right)^{\frac{1}{2}+\varepsilon}}{N(\gamma)^{1-\varepsilon}}\right)
\end{aligned}$$

with the O-constant depending only on ε.

Lemma 5. Let a, a_0, b, b_0 be complex-valued functions over the ring of Gaussian integers,$\left|a_0\left(w\right)\right|,\ \left|b_0\left(w\right)\right| \leq 1$. Let

$$A\left(w\right) = \sum_{\substack{\delta\in G\\ a\left(\delta\right)=w}} a_0\left(\delta\right)$$

$$B\left(w\right)=\sum_{\substack{\delta\in G\\ b\left(\delta\right)=w}} b_0\left(\delta\right)$$

and let $A\left(w\right)$,$\mathrm{B}\left(w\right)<<N\left(w\right)^{\varepsilon}$, $\varepsilon>0$.
Let us assume for α,$\gamma\in G$ and $Res>1$

$$f\left(w;\alpha,\gamma\right)=\sum_{\substack{w_1,w_2=w\\ w_2\equiv\alpha\left(mod\gamma\right)}} A\left(w_1\right)\mathrm{B}\left(w_2\right)$$

$$F\left(s\right)=\sum_{w} f\left(w;\alpha,\gamma\right)N\left(w\right)^{-s}.$$

Then for $c\geqslant 1+\varepsilon$, $T>1$, $1<N\left(\gamma\right)\leqslant x$ we have

$$\begin{aligned}&\textstyle\sum_{N(w)\leqslant x} f\left(w;\alpha,\gamma\right)=\\ &=\tfrac{1}{2\pi i}\int_{c-iT}^{c+iT}\left[F\left(s\right)-\sum_{\beta\in\mathrm{B}}\mathrm{B}\left(\alpha+\beta\gamma\right)N^{-s}\left(\alpha+\beta\gamma\right)\sum_{w}\mathrm{A}\left(w\right)N\left(w\right)^{-s}\right]\tfrac{x^s}{s}ds+\\ &+\sum_{\beta\in\mathrm{B}}\mathrm{B}\left(\alpha+\beta\gamma\right)\sum_{w\in\Omega}\mathrm{A}\left(w\right)+\mathrm{O}\left(x^{c+2\varepsilon}T^{-1}\left(c-1\right)^2\min\left\{1,\frac{x^{1/2}}{N(\gamma)^{3/2}}\right\}\right)+\\ &+\mathrm{O}\left(\frac{x^{1/2+\varepsilon}}{N^{1/2}(\gamma)}\log T\right)\end{aligned}\tag{5}$$

where

$$\mathrm{B}=\left\{\beta\left|N\left(\beta\right)=0,1\right.\right\},\Omega(\alpha)=\left\{w(\alpha)\left|N\left(w\right)\leq\frac{x}{N\left(\alpha+\beta\gamma\right)}\right.\right\}.$$

The proofs of Lemma 4 and 5 are similar to proofs of Lemmas 6 and in [8].

Main Results. We denote

$$\tau^{(m)}\left(w;E_n^+\right):=\sum_{\substack{\alpha\in E_n^+\\ \alpha\left|w\right.}} e^{4mi\arg w}=e^{4mi\arg w}\tau^{(0)}\left(w;E_n^+\right)=e^{4mi\arg w}\tau\left(w;E_n^+\right).$$

First we prove the following statement.

Theorem 1. Let p be a prime rational number, $p\equiv 3\left(mod4\right)$. For any positive integer n the asymptotic formula

$$\begin{aligned}&\textstyle\sum_{N(w)\leqslant x}\tau^{(m)}\left(w;E_n^+\right)=\varepsilon_m\left(\tfrac{\pi^2}{2}\tfrac{p+1}{p}\tfrac{x\log x}{p^n}+\right.\\ &+\tfrac{\pi x}{4p^n}\tfrac{p+1}{p}\left(b_0\left(\tilde{\chi}_0\right)+\gamma+L'\left(1,\chi_4\right)\right)+O\left(x^{1/2+\varepsilon}\log T\right)+\\ &\left.+O\left(\left(\tfrac{T^2+m^2}{T^2}\right)^{\frac{1}{2}}p^{-n}x^{\frac{1}{2}}\right)\right)\end{aligned}$$

holds, where the parameter $b_0(\tilde{\chi}_0)$ determine in (**??**)

$$\varepsilon_m = \begin{cases} 1\; if\, m = 0 \\ 0\; if\, m \neq 0 \end{cases}.$$

a parameter $b_0(\tilde{\chi}_0)$ determine in (9) (see the bellow).
Proof. For $m = 0$ we use Lemma 5 with $a(w) = b(w) = w$, $a_0(w) = 1$

$$b_0(w) = \begin{cases} 1\;\; if\, w \in E_n^+ \\ \;\; 0\;\; else \end{cases}$$

For $Res > 1$ and $\alpha \in E_n^+$ we have:

$$p^{-2ns}\zeta_0(s;0{,}0)\,\zeta_0\left(s;\frac{\alpha}{p^n},0\right) = \sum_w \tau^*\left(w;\alpha,E_n^+\right) N(w)^{-s},$$

where

$$\tau\left(w;\alpha,E_n^+\right) = \sum_{\substack{w = w_1 w_2 \\ w_2 \equiv \alpha\,(p^n) \\ \alpha \in E_n^+}} 1.$$

Hence, by (5),

$$\begin{aligned} &\textstyle\sum_{N(w)\leqslant x} \tau\left(w;\alpha,E_n^+\right) = \\ &= \tfrac{1}{2\pi i}\textstyle\int_{c-iT}^{c+iT} \zeta_0(s;0{,}0)\left[p^{-2ns}\zeta_0\left(s;\tfrac{\alpha}{p^n},0\right) - \sum_{\beta\in\mathrm{B}} N\left(\alpha+\beta p^n\right)^{-s}\right]\tfrac{x^s}{s}ds + \\ &+ \textstyle\sum_{\mathrm{B}}\sum_{\Omega(\alpha)} 1 + \mathrm{O}\left(\tfrac{x^{c-2\varepsilon}}{T(c-1)^2}\min\left\{1,\left(\tfrac{x}{p^{3n}}\right)^{1/2}\right\}\right) + \mathrm{O}\left(x^{1/2+\varepsilon}p^{-n}\log T\right). \end{aligned} \tag{6}$$

In order to estimate the integral in (6) we take $c = 1+3\varepsilon$ and use the residual theorem. Thus Lemmas 1 and 2 give

$$\begin{aligned} &\tfrac{1}{2\pi i}\textstyle\int_{c-iT}^{c+iT} \zeta_0(s;0{,}0)\left[\zeta_0\left(s;\tfrac{\alpha}{p^n},0\right) - \sum_{\beta\in\mathrm{B}} N\left(\alpha+\beta p^n\right)^{-s}\right]\left(\tfrac{x}{p^{2n}}\right)^s\tfrac{ds}{s} = \\ &= \underset{s=1}{res}\left\{\zeta_0(s)\left[\zeta_0\left(s;\tfrac{\alpha}{p^n},0\right) - \textstyle\sum_{\mathrm{B}} N\left(\tfrac{\alpha}{p^n}+\beta\right)^{-s}\right]\left(\tfrac{x}{p^{2n}}\right)^s\tfrac{1}{s}\right\} + \\ &+ \textstyle\int_{1/2-iT}^{1/2+iT} \zeta_0(s;0{,}0)\left[\zeta_0\left(s;\tfrac{\alpha}{p^n},0\right) - \sum_{\beta\in\mathrm{B}} N\left(\alpha+\beta p^n\right)^{-s}\right]\left(\tfrac{x}{p^{2n}}\right)^s\tfrac{ds}{s} + \\ &+ \mathrm{O}\left(x^{1+3\varepsilon}T^{-1}p^{-2n}\right) + \mathrm{O}\left(\left(xp^{-2n}\right)^{\frac{1}{2}+\varepsilon}\right) \qquad (\mathbf{??}) \end{aligned}$$

Applying Cauchy inequality in an integral in (**??**), we obtain

$$\begin{aligned} I = &\left|\textstyle\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \zeta_0(s;0{,}0)\left[\zeta_0\left(s;\tfrac{\alpha}{p^n},0\right) - \sum_{\mathrm{B}} N\left(\tfrac{\alpha}{p^n}+\beta\right)^{-s}\right]\left(\tfrac{x}{p^{2n}}\right)^s\tfrac{ds}{s}\right| \leqslant \\ &\leqslant x^{1/2}\left(\textstyle\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \left|\zeta_0(s;0{,}0)\right|^2\tfrac{ds}{s}\right)^{1/2}\left(\textstyle\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\left|p^{-2ns}\zeta_0\left(s;\tfrac{\alpha}{p^n},0\right) - \sum_{\mathrm{B}}\tfrac{1}{N(\alpha+\beta p^n)}\right|^2\tfrac{|ds|}{|s|}\right)^{1/2} \end{aligned}$$

Since $\zeta_0(s;0{,}0) = 4\xi(s)\,L(s,\chi_4)$ we infer

$$\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} |\zeta_0(s;0{,}0)|^2 \frac{|ds|}{|s|} << \left(\int_1^T |\xi(s)|^4 \frac{dt}{t}\right)^{\frac{1}{2}} \left(\int_1^T |L(s,\chi_4)|^4 \frac{dt}{t}\right)^{\frac{1}{2}} << \log^5 T$$

(we use estimates of mean value for fourth moment of $\xi(s)$ and $L(s,\chi_4)$, see [2]). Moreover, Lemma 4 gives

$$\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \left|p^{-2ns}\zeta_0\left(s;\frac{\alpha}{p^n},0\right) - \sum_{\mathrm{B}} N(\alpha+\beta p^n)^{-s}\right|^2 \frac{|ds|}{|s|} << \frac{T^{\varepsilon}}{p^{2n(1-\varepsilon)}}$$

Therefore,

$$I << x^{1/2} p^{-n+\varepsilon} T^{\varepsilon} << x^{\frac{1}{2}+\varepsilon} T^{1+\varepsilon} p^{-n}.$$

We take

$$T = \begin{cases} x^{1/2} & if\ \ p^n \le x^{1/3} \\ x^{1/4} p^{n/2} & if\ \ p^n > x^{1/3} \end{cases}.$$

Hence,

$$\begin{aligned} &\textstyle\sum_{N(w)\le x} \tau(w;\alpha,E_n^+) = \\ &= \underset{s=1}{res}\left\{\zeta_0(s;0{,}0)\left[\zeta_0\left(s;\tfrac{\alpha}{\gamma},0\right) - \textstyle\sum_{\mathrm{B}} N\left(\tfrac{\alpha}{p^n}+\beta\right)^{-s}\right]\left(\tfrac{x}{p^{2n}}\right)^s \tfrac{1}{s}\right\} + \\ &+ \textstyle\sum_{\beta\in\mathrm{B}}\sum_{w\in\Omega(\alpha)} 1 + \mathrm{O}\left(x^{1/2+3\varepsilon} p^{-n}\right). \end{aligned} \qquad (8)$$

But we have

$$\sum_{\alpha\in\mathrm{E}_n^+}\sum_{\beta\in\mathrm{B}}\sum_{w\in\Omega(\alpha)} 1 = \pi x \sum_{\alpha\in\mathrm{E}_n^+}\sum_{\mathrm{B}} N(\alpha+\beta p^n)^{-1} + \mathrm{O}\left(\left(\frac{x}{N(\alpha)}\right)^{1/3}\right) \qquad (6)$$

$$\sum_{\alpha}\underset{s=1}{res}\left\{-\zeta_0(s)\sum_{\mathrm{B}}\frac{x^s}{sN(\alpha+\beta p^n)}\right\} = -\pi x\sum_{\alpha}\sum_{\mathrm{B}}\frac{1}{N(\alpha+\beta p^n)} \qquad (7)$$

$$\begin{aligned} &\textstyle\sum_{\alpha}\underset{s=1}{res}\left\{\zeta_0(s;0{,}0)\,\zeta_0\left(s;\tfrac{\alpha}{p^n},0\right)p^{-2ns}\tfrac{x^s}{s}\right\} = \\ &= \underset{s=1}{res}\left[\left(\tfrac{\pi}{4}\tfrac{1}{s-1} + \left(\tfrac{\pi\gamma}{4} + L'(1,\chi_4) + \ldots\right)\right)\times\right. \\ &\left.\times\textstyle\sum_{\alpha\in E_n^+}\tfrac{1}{\left|E_n^+\right|}\tfrac{1}{p^{2ns}}\left(\textstyle\sum_{\hat\chi\in\hat E_n^+}\hat\chi\left(\alpha^{-1}\right)\zeta(s,\hat\chi)\right)\tfrac{x^s}{s}\right] \end{aligned} \qquad (8)$$

where $\hat{\mathrm{E}}_n^+$ is the group of characters for E_n^+.

$$\zeta\left(s,\hat{\chi}\right)=\sum_{w\in G}\frac{\tilde{\chi}\left(w\right)}{N\left(w\right)^{s}},\tilde{\chi}\in\hat{E}_{n}^{+}.$$

$$\zeta\left(s,\hat{\chi}\right)=\frac{\varepsilon\left(\hat{\chi}\right)}{s-1}+b_{0}\left(\hat{\chi}\right)+b_{1}\left(\hat{\chi}\right)\left(s-1\right)+...$$

$$\varepsilon\left(\tilde{\chi}\right)=\begin{cases}\frac{\pi}{4}\frac{p^{2}-1}{p^{2}} & if\ \ \tilde{\chi}=\tilde{\chi}_{0}\ is\ the\ principal\ character\ from\ \hat{E}_{n}^{+};\\ 0 & else.\end{cases}$$

$$b_{0}\left(\tilde{\chi}_{0}\right)=\frac{\pi}{4}\frac{p^{2}-1}{p^{2}}\left(\gamma+\frac{L'\left(1,\chi_{4}\right)}{L\left(1,\chi_{4}\right)}+\frac{\log p^{2}}{p^{2}-1}\right)\tag{9}$$

Hence,

$$\begin{array}{l}\underset{s=1}{res}\left\{\sum_{\alpha\in E_{n}^{+}}\zeta_{0}\left(s;0,0\right)\zeta_{0}\left(s;\frac{\alpha}{p^{n}},0\right)p^{-2ns}\frac{x^{s}}{s}\right\}=\\ =\frac{\pi^{2}}{2}\frac{p+1}{p}\frac{x\log x}{p^{n}}+\frac{\pi x}{4p^{n}}\frac{p+1}{p}\left(b_{0}\left(\tilde{\chi}_{0}\right)+\gamma+L'\left(1,\chi_{4}\right)\right)\end{array}\tag{10}$$

■

In the case $m\neq 0$ the proofs follows by analogous if we take $T=\left(xp^{-2n}\right)^{\frac{1}{2}-2\varepsilon}$.

Well known lemma of Vinogradov on approximation of characteristic function of segment $\Delta\subset[0,1)$ by truncated Fourie series gives the main result our paper.

Theorem 2. For $p^{n}\ll x^{\frac{1}{2}-\varepsilon}$ and $(\phi_{2}-\phi_{1})\ll\left(xp^{-2n}\right)^{-\frac{1}{2}+\varepsilon}$ the following asymptotic formula

$$\sum_{\substack{\alpha\in E_{n}^{+}\\ \varphi_{1}\leqslant\arg w\leqslant\varphi_{2}\\ N(w)\leqslant x}}\tau\left(w;\alpha,E_{n}^{+}\right)=\left(\frac{\pi^{2}x\log x}{p^{n}}\cdot\frac{p+1}{p}+\frac{x}{p^{2n}}\sum_{\alpha\in E_{n}}c\left(\alpha,p^{n}\right)\right)(\varphi_{2}-\varphi_{1})+O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

holds.

The constant in symbol "O" depends only on ε, $\varepsilon>0$.

Concluding remark. The Theorem 2 establishes that the divisor function is an Gaussian integers with divisors from the norm group E_n^+.

Conclusion. The divisor function in norm group E_n^+ was investigated. The asymptotic formula is obtained.

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Дільники Гаусових чисел на норменій підгрупі E_n^+

Резюме

Розглядається функція дільників на норменій підгрупі E_n^+, де множина E_n^+ – мультиплікативна підгрупа мультиплікативної групи класів лишків модулю p^n над $\mathbb{Z}[i]$. Побудована асимптотична формула.
Ключові слова: гаусові числа, функція дільників, асимптотична формула.

Радова А. С.
Делители Гауссовых чисел на норменной подгруппе E_n^+

Резюме

Рассматривается функция делителей на норменной подгруппе E_n^+, где множество E_n^+ – мультипликативная подгруппа мультипликативной группы классов вычетов по модулю p^n над $\mathbb{Z}[i]$. Построена асимптотическая формула.
Ключевые слова: гауссовы числа, функция делителей, асимптотическая формула.