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ASYMPTOTIC REPRESENTATIONS OF ONE CLASS OF SOLUTIONS WITH SLOWLY VARYING DERIVATIVES TO NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

For essentially nonlinear differential equations, that are in some sense close to equations of Emden-Fowler type necessary and sufficient conditions of existence of one class of critical solutions were found. Asymptotic representations for such solutions and their first order derivatives are also established.
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Introduction

Differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y')\exp(R(\ln|y|, \ln|yy'||)), \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\to]0, +\infty[$ $(-\infty < a < \omega \leq +\infty)$, $\varphi_i : \Delta_{Y_i} \to]0, +\infty[$ are continuous functions, $R :]0; +\infty[\to]0; +\infty[$ $R :]0; +\infty[\times]0; +\infty[\to]0, +\infty[$ is continuously differentiable function. Here $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is either the interval $[y_i^0; Y_i[,^*$ or the interval $]Y_i; y_i^0]$ $(i = 0, 1)$. Moreover, it is supposed, that every function φ_i $(i = 0, 1)$ is regularly varying of the order σ_i ([1], chapter 1, §1.1, p. 9) as the argument tends to Y_i and $\sigma_0 + \sigma_1 \neq 1$.

Moreover, suppose the function R satisfy the conditions

$$\lim_{(y_0,y_1)\to(+\infty,+\infty)} R(y_0, y_1) = +\infty, \tag{2}$$

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$$\lim_{y_i\to+\infty}\frac{y_i\dfrac{\partial R}{\partial y_i}(y_0,y_1)}{R(y_0,y_1)}=\gamma_i,\ \text{uniformly by } y_j \quad i,j\in\{0,1\},\ i\neq j, \tag{3}$$

where $0<\gamma_0+\gamma_1<1,\ \gamma_0\neq 0$.

Definition 1. The solution y of equation (1) is called $P_\omega(Y_0,Y_1,\lambda_0)$ if it is defined on $[t_0,\omega[\subset[a,\omega[$ and

$$\lim_{t\uparrow\omega}y^{(i)}(t)=Y_i \quad (i=0,1), \quad \lim_{t\uparrow\omega}\frac{(y'(t))^2}{y(t)y''(t)}=\lambda_0. \tag{4}$$

Let the function $\varphi:\Delta_Y\to]0,+\infty[$ be a regularly varying function of the order σ as $z\to Y$ $(z\in\Delta_Y)$ $(Y\in\{0,\infty\}$, Δ_Y is a onesided neighborhood of $Y)$. We say that function φ satisfies the condition S if for any slowly varying as $z\to Y \quad (z\in\Delta_Y)$ function $L:\Delta_{Y_i}\to]0;+\infty[$, such, that

$$\lim_{\substack{z\to Y\\ z\in\Delta_Y}}\frac{zL'(z)}{L(z)}=0,$$

the following equality takes place

$$\Theta(zL(z))=\Theta(z)(1+o(1)) \quad \text{as } z\to Y, \quad (z\in\Delta_Y),$$

where $\Theta(z)=\varphi(z)|z|^{-\sigma}$.

Some classes of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of the equation (1) were investigated before (look, for example, [4]). A class of $P_\omega(Y_0,Y_1,\pm\infty)$-solutions of equations of the type (1) has been considered firstly for cases $R(z)\equiv 0$ and $\varphi_0(z)|z|^{-\sigma_0}$ satisfies the condition S. Later it has turned out to extend the results on more general cases (look, for example,[3]). But general case of equation (1), hasn't been considered before. Let us notice, that the derivative of every $P_\omega(Y_0,Y_1,\pm\infty)$-solution is a slowly varying function as $t\uparrow\omega$. It makes additional difficulties by the investigations.

Main Results

Let introduce subsidiary notations

$$\pi_\omega(t)=\begin{cases} t & \text{as } \omega=+\infty,\\ t-\omega & \text{as } \omega<+\infty,\end{cases} \qquad \Theta_i(z)=\varphi_i(z)|z|^{-\sigma_i}, \quad (i=0,1)$$

*If $Y_i=+\infty (Y_i=-\infty)$ we respectively suppose, that $y_i^0>0$ $(y_i^0<0)$.

and in case $\lim\limits_{t\uparrow\omega}|\pi_\omega(t)|\mathrm{sign}y_0^0=Y_0$,

$$N(t)=\alpha_0 p(t)|\pi_\omega(t)|^{\sigma_0+1}\Theta_0(|\pi_\omega(t)|\mathrm{sign}y_0^0) \text{ as } t\in[b,\omega[,$$

$$I_0(t)=\alpha_0\int\limits_{A_\omega^0}^{t} p(\tau)|\pi_\omega(\tau)|^{\sigma_0}\Theta_0(|\pi_\omega(\tau)|\mathrm{sign}y_0^0)d\tau,$$

$$A_\omega^0=\begin{cases} b, \text{as } \int\limits_b^\omega p(\tau)|\pi_\omega(\tau)|^{\sigma_0}\Theta_0(|\pi_\omega(\tau)|\mathrm{sign}y_0^0)d\tau=+\infty,\\ \omega, \text{as } \int\limits_b^\omega p(\tau)|\pi_\omega(\tau)|^{\sigma_0}\Theta_0(|\pi_\omega(\tau)|\mathrm{sign}y_0^0)d\tau<+\infty.\end{cases}$$

Here we choose $b\in[a,\omega[$ by such a way, that $|\pi_\omega(t)|\mathrm{sign}y_0^0\in\Delta_0$ as $t\in[b,\omega[$.

Theorem 1. *The conditions*

$$Y_0=\begin{cases}\pm\infty, & as\ \omega=+\infty,\\ \\ 0, & as\ \omega<+\infty,\end{cases}\qquad \pi_\omega(t)y_0^0y_1^0>0 \ as\ t\in[a;\omega[. \tag{5}$$

are necessary for the existence of $P_\omega(Y_0,Y_1,\pm\infty)$*-solutions of the equation* (1). *If function* φ_0 *satisfies the condition* S *and*

$$\lim_{t\uparrow\omega}\frac{I_0(t)\sum\limits_{i=0}^{1}\gamma_i|\ln|\pi_\omega(t)||^{\gamma_i-1}\frac{\partial R}{\partial y_i}(|\ln|\pi_\omega(t)||,|\ln|\pi_\omega(t)||)}{\pi_\omega(t)I_0'(t)}=0, \tag{6}$$

then conditions

$$\begin{gathered} y_1^0 I_0(t)(1-\sigma_0-\sigma_1)>0 \quad as\ t\in[a,\omega[,\\ \lim_{t\uparrow\omega}y_1^0|I_0(t)|^{\frac{1}{1-\sigma_0-\sigma_1}}=Y_1,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)I_0'(t)}{I_0(t)}=0.\end{gathered} \tag{7}$$

together with conditions (5) *are necessary and sufficient for the existence of the mentioned solutions of the equation* (1). *Moreover, for each* $P_\omega(Y_0,Y_1,\pm\infty)$*-solution of the equation* (1) *the next asymptotic representations take place as* $t\uparrow\omega$

$$\begin{gathered}\frac{y'(t)|y'(t)|^{-\sigma_0}}{\varphi_1(y'(t))\exp(R(\ln|y|,\ln|yy'||))}=(1-\sigma_0-\sigma_1)I_0(t)[1+o(1)],\\ \frac{y'(t)}{y(t)}=\frac{1}{\pi_\omega(t)}[1+o(1)].\end{gathered} \tag{8}$$

Proof. *The necessity.*

Let function $y : [t_0, \omega[\to \Delta_{Y_0}$ is a $P_\omega(Y_0, Y_1, \pm\infty)-$solution of the equation (1). By (4) it follows from the equality

$$\frac{y''(t)y(t)}{(y'(t))^2} = \left(\frac{y'(t)}{y(t)}\right)' \cdot \left(\frac{y'(t)}{y(t)}\right)^{-2} + 1,$$

that

$$\left(\frac{y'(t)}{y(t)}\right)' \cdot \left(\frac{y'(t)}{y(t)}\right)^{-2} = -1 + o(1) \text{ as } t \uparrow \omega.$$

From this by (4) the next asymptotic representations take place

$$y(t) = \pi_\omega(t)y'(t)[1 + 0(1)], \qquad y''(t) = o\left(\frac{y'(t)}{\pi_\omega(t)}\right) \text{ as } t \uparrow \omega. \tag{9}$$

From the first formula we get the first one of representations (8) and condition (5). It also follows from (9), that there exists such a slowly varying continuously differentiable function $L : \Delta_{Y_0} \to]0, +\infty[$, that $y(t) = \pi_\omega(t)L(\pi_\omega(t))$. By condition S we obtain $\Theta_0(y(t)) = \Theta_0(|\pi_\omega(t)|\text{sign} y_0^0)[1 + o(1)]$ as $t \uparrow \omega$.

Let us rewrite (1) in the form

$$\frac{y''(t)}{\varphi_1(y'(t))|y'(t)|^{\sigma_0}} = I_0'(t)\exp(R(\ln|y|, \ln|yy'||))[1 + o(1)] \text{ as } t \uparrow \omega. \tag{10}$$

Now let us suppose the condition (6) takes place and denote

$$\lim_{t\uparrow\omega} I_0(t) = J_0.$$

By conditions (6) and (9) the function $\exp(R(|\ln|y(I_0^{-1}(z))||, |\ln|y(I_0^{-1}(z))y'(I_0^{-1}(z))||))$ is a slowly varying function as $z \to J_0$. Here I_0^{-1} is the inverse function for I_0.

Therefore, using (10) we get

$$\frac{y'(t)|y'(t)|^{-\sigma_0}}{\varphi_1(y'(t))\exp(R(|\ln|y(t)||, |\ln|y(t)y'(t)||))} = (1-\sigma_0-\sigma_1)I_0(t)[1+o(1)] \text{ as } t \uparrow \omega. \tag{11}$$

So the representation (8) is grounded. Taking into account sign of the function $y'(t)$ we obtain the first and the second of conditions (7). Using the second of relations (9) we have by (11) and (10), that

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)I_0'(t)\varphi_1(y'(t))}{|y'(t)|^{1-\sigma_0}} = 0.$$

The third of conditions (7) follows from this relation. And thus the necessity is proved.

The sufficiency. Let's suppose, that the function φ_1 satisfies condition S and conditions (5)-(7) of the theorem take place. Let's denote

$$g(v_0, v_1) = \exp(R(|\ln|v_0||, |\ln|v_0 v_1||))L_1(v_1),$$

where $L_1 : \Delta_{Y_1} \to]0, +\infty[$ is such slowly varying function as $z \to Y_1 (z \in \Delta_{Y_1})$, that

$$L_1(z) = \Theta_1(z)[1 + o(1)] \text{ as } z \to Y_1 \quad (z \in \Delta_{Y_1}), \qquad \lim_{\substack{z \to Y_i \\ z \in \Delta_{Y_1}}} \frac{z L_1'(z)}{L_1(z)} = 0. \tag{12}$$

According to properties of the function R and (12), we get

$$\lim_{\substack{v_i \to Y_i \\ v_i \in \Delta_{Y_i}}} \frac{v_i \frac{\partial g}{\partial v_i}(v_0, v_1)}{g(v_0, v_1)} = 0 \text{ uniformly by } v_j \in \Delta_{Y_j},\ j \neq i,\ i, j = 0, 1. \tag{13}$$

So, we can take $\tilde{\Delta}_{Y_i} \subset \Delta_{Y_i}$ $(i = 0, 1)$ in such a form, that

$$\left| \frac{v_i \frac{\partial g}{\partial v_i}(v_0, v_1)}{g(v_0, v_1)} \right| < \zeta,\ (i = 0, 1) \text{ as } (v_0, v_1) \in \tilde{\Delta}_{Y_0} \times \tilde{\Delta}_{Y_1}. \tag{14}$$

Here $0 < \zeta < \frac{|1 - \sigma_0 - \sigma_1|}{8}$, ζ is sufficiently small and

$$\tilde{\Delta}_{Y_i} = \begin{cases} \{[\tilde{y}_i^0, Y_i[, \text{ if } \Delta_{Y_i} = [y_i^0, Y_i[, \quad y_i^0 \leq \tilde{y}_i^0 < Y_i; \\ \\ ]Y_i, \tilde{y}_i^0], \text{ if } \Delta_{Y_i} =]Y_i, y_i^0], \quad Y_i > \tilde{y}_i^0 \geq y_i^0, \end{cases} \qquad i = 0, 1.$$

Let's consider the function

$$F(s_0, s_1) = \begin{pmatrix} \dfrac{|s_1|^{1 - \sigma_0 - \sigma_1}}{g(s_0, s_1)} \\ \\ \dfrac{s_1}{s_0} \end{pmatrix}$$

on the set $\tilde{\Delta}_{Y_0} \times \tilde{\Delta}_{Y_1}$.

Using (13) we get

$$\lim_{\substack{s_1 \to Y_1 \\ s_1 \in \tilde{\Delta}_{Y_1}}} \frac{|s_1|^{1 - \sigma_0 - \sigma_1}}{g(s_0, s_1)} = \Upsilon, \qquad \text{uniformly by } s_0 \in \tilde{\Delta}_{Y_0},$$

$$\Upsilon = \begin{cases} +\infty, \text{ if } Y_1 = \infty \text{ and } 1-\sigma_0-\sigma_1 > 0, \\ \ \text{or } Y_1 = 0 \text{ and } 1-\sigma_0-\sigma_1 < 0, \\ \\ 0, \text{ if } Y_1 = \infty \text{ and } 1-\sigma_0-\sigma_1 < 0, \\ \ \text{or } Y_0 = 0 \text{ and } 1-\sigma_0-\sigma_1 > 0. \end{cases}$$

Let us show, that F sets one to one correspondence between the set $\tilde{\Delta}_{Y_0} \times \tilde{\Delta}_{Y_1}$ and the set

$$F(\tilde{\Delta}_{Y_0} \times \tilde{\Delta}_{Y_1}) = \begin{cases} \left[\frac{|\tilde{y}_0^1|^{1-\sigma_0-\sigma_1}}{g(\tilde{y}_0^0, \tilde{y}_0^1)}; \Upsilon\right) \times \Delta_0, & \text{as } \frac{|\tilde{y}_0^1|^{1-\sigma_0-\sigma_1}}{g(\tilde{y}_0^0, \tilde{y}_0^1)} < \Upsilon, \\ \\ \left(\Upsilon; \frac{|\tilde{y}_0^1|^{1-\sigma_0-\sigma_1}}{g(\tilde{y}_0^0, \tilde{y}_0^1)}\right] \times \Delta_0, & \text{as } \frac{|\tilde{y}_0^1|^{1-\sigma_0-\sigma_1}}{g(\tilde{y}_0^0, \tilde{y}_0^1)} > \Upsilon. \end{cases} \tag{15}$$

Here

$$\Delta_0 = \begin{cases} [\frac{\tilde{y}_0^1}{\tilde{y}_0^0}; Y_0^0), & \text{as } \lambda_0 < 0, \frac{\tilde{y}_0^1}{\tilde{y}_0^0} < Y_0^0, \\ \\ (Y_0^0; \frac{\tilde{y}_0^1}{\tilde{y}_0^0}], & \text{as } \lambda_0 < 0, \frac{\tilde{y}_0^1}{\tilde{y}_0^0} > Y_0^0, \end{cases} \tag{16}$$

$$Y_0^0 = \begin{cases} \quad 0, & \text{as } Y_0 = 0, \\ -\infty & \text{as } Y_0 = 0 \text{ and } \omega < +\infty, \\ +\infty & \text{as } Y_0 = 0 \text{ and } \omega = +\infty. \end{cases}$$

Let's consider the behavior of the function $\frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(s_0,s_1)}$ on straight lines

$$s_0 = ks_1, \qquad k \in \mathbb{R} \setminus \{0\}. \tag{17}$$

On every such line we have $\dfrac{|s_1|^{1-\sigma_0-\sigma_1}}{g(s_0,s_1)} = \dfrac{|s_1|^{1-\sigma_0-\sigma_1}}{g(ks_1,s_1)}$. Taking into account (14) we obtain, that

$$\operatorname{sign}\left(\frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(ks_1,s_1)}\right)'_{s_1} = \operatorname{sign}(y_1^0(1-\sigma_0-\sigma_1)).$$

Therefore the function $\dfrac{|s_1|^{1-\sigma_0-\sigma_1}}{g(ks_1,s_1)}$ is strongly monotone on every line of the type (17). Let suppose, that the correspondence F is not a one to one type. Then

$$\exists (p_0,p_1),(q_0,q_1) \in \tilde{\Delta}_{Y_0} \times \tilde{\Delta}_{Y_1}, \quad (p_0,p_1) \neq (q_0,q_1): \qquad F(p_0,p_1) = F(q_0,q_1).$$

Taking into account the definitions of the sets $\tilde{\Delta}_{Y_0}, \tilde{\Delta}_{Y_1}$, the last equality means, that

$$\frac{|p_1|^{1-\sigma_0-\sigma_1}}{g(p_0,p_1)} = \frac{|q_1|^{1-\sigma_0-\sigma_1}}{g(q_0,q_1)}, \quad \frac{p_0}{p_1} = \frac{q_0}{q_1} = c \in \mathbb{R} \setminus \{0\}. \tag{18}$$

So, the points (p_0, p_1) and (q_0, q_1) lie on a line of the type (17). But in this case equalities (18) can not take place, because the function $\frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(s_1, cs_1)}$ is strongly monotone on the line. Therefore, there exists the inverse function $F^{-1} : F(\tilde{\Delta}_{Y_0} \times \tilde{\Delta}_{Y_1}) \to \tilde{\Delta}_{Y_0} \times \tilde{\Delta}_{Y_1}$. Taking into account the character of the function F, we have

$$F^{-1}(w_0, w_1) = \begin{pmatrix} F_1^{-1}(w_0,w_1) \\ \\ F_0^{-1}(w_0,w_1) \end{pmatrix} = \begin{pmatrix} F_1^{-1}(w_0,w_1) \\ \\ \frac{1}{w_0} F_1^{-1}(w_0,w_1) \end{pmatrix}.$$

Since by (14) Jakobian of the function F is different from zero as $(s_0, s_1) \in \tilde{\Delta}_{Y_0} \times \tilde{\Delta}_{Y_1}$, the function F^{-1} is continuously differentiable on $F(\tilde{\Delta}_{Y_0} \times \tilde{\Delta}_{Y_1})$.

Let

$$\begin{cases} \dfrac{|y'(t)|^{1-\sigma_0}}{\varphi_1(y') \exp(R(|\ln|y(t)||, |\ln|y(t)y'(t)||))} = (1-\sigma_0-\sigma_1) I_0(t) \operatorname{sign}(y')[1+z_1(x)], \\ \dfrac{y'(t)}{y(t)} = \dfrac{1}{\pi_\omega(t)}[1+z_2(x)], \end{cases} \tag{19}$$

where

$$x = \beta \ln|\pi_\omega(t)|, \qquad \beta = \begin{cases} 1 & \text{as } \omega = +\infty, \\ -1 & \text{as } \omega < \infty, \end{cases}$$

we can reduce the equation (1) to the system

$$\begin{cases} z_1' = \beta G_0(x)[1+z_1] \left(\left(1 - \sigma_0 - \sigma_1 - \dfrac{\Psi_1(x,z_1,z_2) L_1'(\Psi_1(x,z_1,z_2))}{L_1(\Psi_1(x,z_1,z_2))} \right) \times \right. \\ \left. \times \dfrac{K_1(x,z_1,z_2)}{[1+z_1]|1+z_2|^{\sigma_0}} - K_2(x,z_1,z_2) \dfrac{V(x))}{G_0(x)} \left(1 + \dfrac{K_1(x,z_1,z_2) G_0(x)}{[1+z_1][1+z_2]^{\sigma_0-1}} \right) - 1 \right), \\ \\ z_2' = \beta[1+z_2] \left(\dfrac{G_0(x) K_1(x,z_1,z_2)}{(1-\sigma_0-\sigma_1)[1+z_1][1+z_2]^{\sigma_0}} - z_2 \right), \end{cases} \tag{20}$$

where

$$\Psi_0(x,z_1,z_2) = F_0^{-1} \left((1-\sigma_0-\sigma_1) I_0(t(X))[1+z_1(x)], \frac{1}{\pi_\omega(t(x))}[1+z_2(x)] \right),$$

$$\Psi_1(x,z_1,z_2)=F_1^{-1}\left((1-\sigma_0-\sigma_1)I_0(t(x))[1+z_1(x)],\frac{1}{\pi_\omega(t(x))}[1+z_2(x)]\right),$$

$$V(x)=\sum_{i=0}^{1}\gamma_i|\ln|\pi_\omega(t)||^{\gamma_i-1}\frac{\partial R}{\partial y_i}(|\ln|\pi_\omega(t)||,|\ln|\pi_\omega(t)||)$$

$$G_0(x)=\frac{\pi_\omega(t(x))I_0'(t(x))}{I_0(t(x))},$$

$$K_1(x,z_1,z_2)=\frac{\Theta_0(\Psi_0(t(x),z_1,z_2))}{(1-\sigma_0-\sigma_1)\Theta_0(|\pi_\omega(t(x))|\mathrm{sign}y_0^0)},$$

$$K_2(x,z_1,z_2)=$$

$$=\frac{\sum\limits_{i=0}^{1}\gamma_i|\Psi_i(t(x),z_1,z_2)|^{\gamma_i-1}\frac{\partial R}{\partial y_i}(|\ln|\Psi_0(t(x),z_1,z_2))||,|\ln|\Psi_0(t(x),z_1,z_2))\Psi_1(t(x),z_1,z_2)||)}{V(x)}.$$

By (5) it is clear, that

$$\lim_{t\uparrow\omega}\frac{1}{\pi_\omega(t)}=Y_1.$$

Moreover, it follows from the first and the second of conditions (7), that

$$\lim_{t\uparrow\omega}(1-\sigma_0-\sigma_1)I_0(t)=\Upsilon.$$

Therefore, we can choose $t_0\in[a,\omega[$ in such a form, that

$$\begin{pmatrix}(1-\sigma_0-\sigma_1)I_0(t)[1+z_1(x)]\\ \\ \dfrac{1}{\pi_\omega(t)}[1+z_2(x)]\end{pmatrix}\in F(\tilde\Delta_{Y_0}\times\tilde\Delta_{Y_1})\ \text{при}\ t\in[t_0,\omega[,|z_i|\le\frac{1}{2},i=1,2.$$

Then we consider the system of differential equations (20) on the set

$$\Omega=[x_0,+\infty[\times D,\ \text{где}\ x_0=\beta\ln|\pi_\omega(t_0)|,$$

$$D=\{(z_1,z_2):|z_i|\le\frac{1}{2},i=1,2\}.$$

Let's rewrite the system in the form

$$\begin{cases}z_1'=G_0(x)(A_{11}z_1+A_{12}z_2+R_1(x,z_1,z_2)+R_2(z_2)),\\ z_2'=A_{21}z_1+A_{22}z_2+R_3(x,z_1,z_2)+R_4(z_2),\end{cases}\tag{21}$$

where

$$A_{11}=A_{22}=-\beta,\quad A_{12}=-\beta\sigma_0,\quad A_{21}=0,$$

$$R_1(x,z_1,z_2)=-\beta[1+z_1]\left(K_2(x,z_1,z_2)\frac{V(x)}{G_0(x)}\left(1+\frac{K_1(x,z_1,z_2)G_0(x)}{(1+z_1)(1+z_2)^{\sigma_0-1}}\right)+\right.$$

$$\left.+\frac{K_1(x,z_1,z_2)}{(1+z_1)|1+z_2|^{\sigma_0}}\frac{\Psi_1(x,z_1,z_2)L_1'(\Psi_1(x,z_1,z_2))}{L_1(\Psi_1(x,z_1,z_2))}\right)+\beta\frac{|K_1(x,z_1,z_2)|(1-\sigma_0-\sigma_1)-1}{|1+z_2|^{\sigma_0}},$$

$$R_2(z_2)=\beta(|1+z_2|^{-\sigma_0}+\sigma_0 z_2),$$

$$R_3(x,z_1,z_2)=\beta\frac{[1+z_2]G_0(x)K_1(x,z_1,z_2)}{(1-\sigma_0-\sigma_1)[1+z_1][1+z_2]^{\sigma_0}},\quad R_4(z_2)=-\beta z_2^2.$$

For $(w_0,w_1)\in F(\tilde\Delta_{Y_0}\times\tilde\Delta_{Y_1})$ the next equality

$$\frac{|F_1^{-1}(w_0,w_1)|^{1-\sigma_0-\sigma_1}}{g(F_0^{-1}(w_0,w_1),F_1^{-1}(w_0,w_1))}=w_1,$$

takes place. Since (14), (5) and the second of conditions (6), it follows from this equality, that as $i=0,1$

$$\lim_{x\to\infty}\Psi_i(t(x),z_1,z_2)=Y_i \text{ uniformly by } (z_1,z_2)\in\left[-\frac{1}{2};\frac{1}{2}\right]\times\left[-\frac{1}{2};\frac{1}{2}\right].$$

Therefore by (12) we have

$$\lim_{x\to\infty}\frac{\Psi_1(t(x),z_1,z_2)L_1'(\Psi_1(t(x),z_1,z_2))}{L_1(\Psi_1(t(x),z_1,z_2))}=0 \text{ uniformly by } (z_1,z_2)\in\left[-\frac{1}{2};\frac{1}{2}\right]\times\left[-\frac{1}{2};\frac{1}{2}\right]. \tag{22}$$

Moreover, it follows from the properties of the function F by conditions (5)-(7), that the function $\Psi_1(t,z_1,z_2)$ is a slowly varying function as $t\uparrow\omega$ uniformly by $(z_1,z_2)\in\left[-\frac{1}{2};\frac{1}{2}\right]\times\left[-\frac{1}{2};\frac{1}{2}\right]$. Since

$$\Psi_0(t,z_1,z_2)=\frac{\pi_\omega(t)\Psi_1(t,z_1,z_2)}{1+z_2},$$

and the function φ_0 together with the logarithmic function satisfy the condition S, we have

$$\lim_{x\to\infty}K_1(x,z_1,z_2)=\frac{1}{1-\sigma_0-\sigma_1} \text{ uniformly by } (z_1,z_2)\in\left[-\frac{1}{2};\frac{1}{2}\right]\times\left[-\frac{1}{2};\frac{1}{2}\right], \tag{23}$$

$$\lim_{x\to\infty}K_2(x,z_1,z_2)=1 \text{ uniformly by } (z_1,z_2)\in\left[-\frac{1}{2};\frac{1}{2}\right]\times\left[-\frac{1}{2};\frac{1}{2}\right]. \tag{24}$$

Since the function R is regularly varying on infinity of the order μ, and $0<\mu<1$, we obtain

$$\lim_{t\uparrow\omega}R'(|\ln|\pi_\omega(t)||)=0. \tag{25}$$

It follows from the third of conditions (7), that

$$\lim_{x\to\infty} G_0(x) = 0. \tag{26}$$

By (6) and (22)–(26) we get as $i = 2, 4$ the next limit relations

$$\lim_{|z_1|+|z_2|\to 0} \frac{R_i(z_2)}{|z_1| + |z_2|} = 0 \text{ uniformly by } x : x \in]x_0, +\infty[,$$

and as $i = 1, 3$

$$\lim_{x\to+\infty} R_i(x, z_1, z_2) = 0 \text{ uniformly by } z_1, z_2 : (z_1, z_2) \in D.$$

By the definition of the function G_0 it is clear, that $\int_{x_0}^{\infty} G_0(x)dx = \infty$.

So, for the system of differential equations (21) all conditions of theorem 2.8 from [2] are fulfilled. According to this theorem, the system (21) has at least one solution $\{z_i\}_{i=1}^2 : [x_1, +\infty[\to \mathbb{R}^2 (x_1 \geq x_0)$, that tends to zero as $x \to +\infty$. By (19) and (20) this solution is corresponded by such solution y of the equation (1), that admit asymptotic representations (8) as $t \uparrow \omega$. By the representations and (1) it is clear that the obtained solution is indeed $P_\omega(Y_0, Y_1, \pm\infty)$-solution. The theorem is proved in a whole.

Conclusion

For differential equations with new class of nonlinearity close regularly varying as arguments tend to critical points, necessary and sufficient conditions of existence of one class of critical solutions were found. Asymptotic representations for such solutions to such equations and their first order derivatives are also established.

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Асимптотичні зображення одного класу розв'язків з повільно змінними похідними для нелінійнихдиференціальних рівнянь другого порядку

Резюме

Для істотно нелінійних диференціальних рівнянь, що є в деякому сенсі близькими до рівнянь типу Емдена-Фаулера було знайдено необхідні та достатні умови існування одного класу особливих розв'язків. Також отримано асимптотичні зображення для таких розв'язків та їх похідних першого порядку.
Ключові слова: диференціальні рівняння, асимптотика, повільно змінні розв'язки, правильно змінні нелінійності, нелінійні диференціальні рівняння, повільно змінні похідні.

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