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ON ASYMPTOTIC REPRESENTATIONS OF ONE CLASS SOLUTIONS OF SECOND-ORDER DIFFERENTIAL EQUATIONS

For the second-order differential equation of general form $y'' = f(t, y, y')$, where $f : [a, \omega[\times\Delta_{Y_0} \times \Delta_{Y_1} \longrightarrow \mathbf{R}$ is continuous function, $-\infty < a < \omega \leq +\infty$, Δ_{Y_i} is a one-neighborhood of Y_i, $Y_i \in \{0, \pm\infty\}$ $(i \in \{0, 1\})$ we study the question of the existence of solutions, for which $\lim\limits_{t\uparrow\omega} y^{(i)}(t) = Y_i$ $(i \in \{0, 1\})$. Among the set of such solutions we separate a sufficiently wide class of so-called $P_\omega(Y_0, Y_1, \lambda_0)$-solutions. Such a solution was previously introduced in the study of the two-term equation $y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y')$, where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\longrightarrow]0, +\infty[$ is continuous function, $\varphi_i : \Delta_{Y_i} \longrightarrow]0, +\infty[$ $(i = 0, 1)$ are continuous regular varying as $z \to Y_i$ $(i = 0, 1)$ functions of orders σ_i $(i = 0, 1)$, such that $\sigma_0 + \sigma_1 \neq 1$. In this paper a condition under which the right-hand side of the equation as $\lambda_0 \in \mathbb{R}\backslash\{0, 1\}$ and $t \uparrow \omega$ in some sense close to the multiplication $\alpha_0 p(t)\varphi_0(y)$ with rapidly varying φ_0 function at $y \to Y_0$, is established. We have obtained necessary and also sufficient conditions of existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions, asymptotic representations of these solutions and their first-order derivative and number of parametric family of these solutions. An example is given.
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Introduction

Consider the differential equation

$$y'' = f(t, y, y'), \tag{1.1}$$

where $f : [a, \omega[\times\Delta_{Y_0} \times \Delta_{Y_1} \longrightarrow \mathbf{R}$ is continuous function, $-\infty < a < \omega \leq +\infty$, Δ_{Y_i} $(i \in \{0, 1\})$ is a one-side neighborhood of Y_i and Y_i $(i \in \{0, 1\})$ is either 0

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or $\pm\infty$. We assume that the numbers μ_i $(i = 0, 1)$ given by the formula

$$\mu_i = \begin{cases} \;\;1 & \text{if eigher } Y_i = +\infty, \quad \text{or} \quad Y_i = 0 \\ & \text{and} \quad \Delta_{Y_i} \quad \text{is right neighborhood of the point } 0, \\ -1 & \text{if eigher } Y_i = -\infty, \quad \text{or} \quad Y_i = 0 \\ & \text{and} \quad \Delta_{Y_i} \quad \text{is left neighborhood of the point } 0, \end{cases}$$

satisfy the relations

$$\mu_0\mu_1 > 0 \quad \text{for} \quad Y_0 = \pm\infty \quad \text{and} \quad \mu_0\mu_1 < 0 \quad \text{for} \quad Y_0 = 0. \tag{1.2}$$

Conditions (1.2) are necessary for the existence of solutions of Eq.(1.1) defined in a left neighborhood of ω and satisfying the conditions

$$y^{(i)}(t) \in \Delta_{Y_i} \qquad \text{for } \; t \in [t_0, \omega[\;\; , \quad \lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \quad (i = 0, 1). \tag{1.3}$$

One of the classes of Eq. (1.1) solutions with properties (1.3) that admits some asymptotic representations is the class of $P_\omega(Y_0, Y_1, \lambda_0)$- solutions.

Definition 1.1. *A solution y of Eq. (1.1) on interval $[t_0, \omega[\subset [a, \omega[$ is called $P_\omega(Y_0, Y_1, \lambda_0)$- solution, where $-\infty \le \lambda_0 \le +\infty$, if, in addition to (1.3), it satisfies the condition*

$$\lim_{t\uparrow\omega} \frac{[y'(t)]^2}{y(t)y''(t)} = \lambda_0.$$

Depending on λ_0 these solutions of Eq.(1.1) have different asymptotic properties (see [1]). For $\lambda_0 \in \mathbb{R} \setminus \{1\}$ in [2] for $f(t, y, y') = \alpha_0 p(t)|y|^{\sigma_0}|y'|^{\sigma_1} \operatorname{sign} y$, where $\alpha_0 \in \{-1, 1\}$,$p : [a, \omega[\longrightarrow]0, +\infty[$ is a continuous function, $\sigma_0 + \sigma_1 \neq 1$, such ratios

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)y'(t)}{y(t)} = \frac{\lambda_0}{\lambda_0 - 1}, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)y''(t)}{y'(t)} = \frac{1}{\lambda_0 - 1},$$

where

$$\pi_\omega(t) = \begin{cases} t & \text{if} \quad \omega = +\infty, \\ t - \omega & \text{if} \quad \omega < +\infty, \end{cases}$$

are established.

Definition 1.2. *We say that a function f satisfies condition $(FN)_{\lambda_0}$ for $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$ if there exist a number $\alpha_0 \in \{-1, 1\}$, a continuous function*

$p : [a, \omega[\longrightarrow]0, +\infty[$ *and twice continuously differentiable function* $\varphi_0 : \Delta_{Y_0} \longrightarrow]0, +\infty[$, *satisfying the conditions*

$$\varphi_0'(y) \neq 0, \quad \lim_{\substack{y \longrightarrow Y_o \\ y \in \Delta_{Y_0}}} \varphi_0(y) = \varphi_0 \in \{0, +\infty\}, \quad \lim_{\substack{y \longrightarrow Y_o \\ y \in \Delta_{Y_0}}} \frac{\varphi_0(y)\varphi_0''(y)}{(\varphi_0'(y))^2} = 1, \tag{1.4}$$

such that, for arbitrary continuously differentiable functions $z_i : [a, \omega[\longrightarrow \Delta_{Y_i}$ $(i = 0, 1)$, *satisfying the conditions*

$$\lim_{t \uparrow \omega} z_i(t) = Y_i \quad (i = 0, 1),$$

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) z_0'(t)}{z_0(t)} = \frac{\lambda_0}{\lambda_0 - 1}, \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t) z_1'(t)}{z_1(t)} = \frac{1}{\lambda_0 - 1},$$

one has representation

$$f(t, z_0(t), z_1(t)) = \alpha_0 p(t) \varphi_0(z_0(t))[1 + o(1)] \quad \textit{as} \quad t \uparrow \omega \tag{1.5}$$

Note that the choice of α_0 and the functions p and φ_0 in definition 1.2 depends on the choice of $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$. It is also obvious that the numbers μ_0, μ_1 determine the signs of any $P_\omega(Y_0, Y_1, \lambda_0)$-solution of Eq. (1.1) and its derivative in a left neighborhood of ω (respectively). Moreover, under condition $(FN)_{\lambda_0}$ sign of second derivative of any $P_\omega(Y_0, Y_1, \lambda_0)$-solution of Eq. (1.1) in a left neighborhood of ω coincides with the value α_0. Then taking into account (1.2), we have

$$\alpha_0 \mu_1 > 0 \quad \text{for} \quad Y_1 = \pm\infty \quad \text{and} \quad \alpha_0 \mu_1 < 0 \quad \text{for} \quad Y_1 = 0. \tag{1.6}$$

Main Results

2. Auxiliary statements

We choose a number $b \in \Delta_{Y_0}$ such that the inequality

$$|b| < 1 \quad \text{for} \quad Y_0 = 0, \quad b > 1 \ (b < -1) \quad \text{for} \quad Y_0 = +\infty \ (Y_0 = -\infty)$$

is respected and put

$$\Delta_{Y_0}(b) = [b, Y_0[\quad \text{if} \quad \Delta_{Y_0} \quad \text{is a left neighborhood of} \quad Y_0,$$
$$\Delta_{Y_0}(b) =]Y_0, b] \quad \text{if} \quad \Delta_{Y_0} \quad \text{is a right neighborhood of} \quad Y_0.$$

Definition 2.1. *Let $f : \Delta_{Y_0} \longrightarrow \mathbb{R} \setminus \{0\}$ be a twice continuously differentiable function. We will say that $f \in \Gamma(Y_0, Z_0)$ if it satisfies the following conditions*

$$f'(y) \neq 0, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} f(y) = Z_0, \quad Z_0 = \left[\begin{array}{l} or \quad 0, \\ eigther \quad \pm\infty, \end{array} \right. \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{f''(y)f(y)}{(f'(y))^2} = 1.$$

First of all, we note that, by virtue of definition 2.1, any function from $\Gamma(Y_0, Z_0)$- class is rapidly varying as $y \to Y_0$.

In [3] using the properties of functions from the class Γ introduced and studied in detail in the monograph Bingham N.H., Goldie C.M., Teugels J.L. [4] (Chapter 3, item 3.10), the following auxiliary assertions about the properties of functions from the class $\Gamma(Y_0, Z_0)$ were established.

Lemma 2.1. *If $f \in \Gamma(Y_0, Z_0)$ then there exists a continuous function $g : \Delta_{Y_0} \longrightarrow \mathbb{R} \setminus \{0\}$, called complementary to f, such that*

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{f(y + ug(y))}{f(y)} = e^u \quad \textit{for any} \quad u \in \mathbb{R},$$

moreover, the complementary function is uniquely determined up to functions equivalent as $y \to Y_0$, for which, for example, one of the following functions

$$\frac{\int\limits_Y^y \left(\int\limits_Y^t f(u)\, du \right) dt}{\int\limits_Y^y f(x)\, dx} \sim \frac{\int\limits_Y^y f(x)\, dx}{f(y)} \sim \frac{f(y)}{f'(y)} \sim \frac{f'(y)}{f''(y)} \quad as \quad y \to Y_0,$$

where

$$Y = \begin{cases} b & if \quad \lim\limits_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} f(y) = \pm\infty, \\ Y_0 & if \quad \lim\limits_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} f(y) = 0, \end{cases}$$

can be chosen.

Lemma 2.2.

1. *If $f \in \Gamma(Y_0, Z_0)$ with complementary function g then* $\lim\limits_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{g(y)}{y} = 0.$

2. *If $f \in \Gamma(Y_0, Z_0)$ with complementary function $g$then for any continuous function $u : \Delta_{Y_0} \longrightarrow \mathbb{R}$that satisfies the conditions*

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} u(y) = u_0 \in \mathbb{R}, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} f(y + u(y)g(y)] = Z_0,$$

there is a limit relation

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{f\left(y+u(y)g(y)\right)}{f(y)} = e^{u_0}.$$

Lemma 2.3. *If $f \in \Gamma(Y_0, Z_0)$ strictly monotone with complementary function gthen its inverse function $f^{-1} : \Delta_{Z_0} \longrightarrow \Delta_{Y_0}$ is slowly varying at $z \to Z_0$ and satisfies the limit relation*

$$\lim_{\substack{z\to Z_0\\ z\in\Delta_{Z_0}}} \frac{f^{-1}(\lambda z) - f^{-1}(z)}{g(f^{-1}(z))} = \ln\lambda \quad \textit{for any} \quad \lambda > 0,$$

moreover for any given $\Lambda > 1$ limit relation is satisfied uniformly in $\lambda \in \left[\frac{1}{\Lambda}, \Lambda\right]$.

Note also, it follows from the Representation Theorem for Γ ([5],Chapter 3, item 3.10, position d) that for a function $f \in \Gamma(Y_0, Z_0)$ there exists a continuously differentiable function $f_1 \in \Gamma(Y_0, Z_0)$ such that

$$\lim_{\substack{y\longrightarrow Y_o\\ y\in\Delta_{Y_0}}} \frac{f(y)}{f_1(y)} = 1 \quad \text{and} \quad \lim_{\substack{y\longrightarrow Y_o\\ y\in\Delta_{Y_0}}} \frac{yf_1(y)}{f_1(y)} = \pm\infty.$$

3. Main results

Now we introduce auxiliary functions and notation as follows:

$$\Phi : \Delta_{Y_0}(b) \longrightarrow \mathbb{R}, \quad \Phi(y) = \int\limits_B^y \frac{ds}{\varphi_0(s)}, \quad B = \begin{cases} b & \text{if} \ \int\limits_b^{Y_0} \frac{ds}{\varphi_0(s)} = \pm\infty, \\ Y_0 & \text{if} \ \int\limits_b^{Y_0} \frac{ds}{\varphi_0(s)} = \text{const}, \end{cases}$$

$$I(t) = \int\limits_A^t \pi_\omega(\tau)p(\tau)\,d\tau, \qquad A = \begin{cases} a & \text{if} \ \int\limits_a^{\omega} \pi_\omega(\tau)p(\tau)\,d\tau = \pm\infty, \\ \omega & \text{if} \ \int\limits_a^{\omega} \pi_\omega(\tau)p(\tau)\,d\tau = \text{const}, \end{cases}$$

$$Y(t) = \Phi^{-1}\left(\alpha_0(\lambda_0 - 1)I(t)\right), \quad H(t) = \frac{Y(t)\varphi_0'(Y(t))}{\varphi_0(Y(t))}, \quad q(t) = \frac{Y(t)'\pi_\omega(t)}{Y(t)},$$

$$k(t) = q(t)\left(\frac{\varphi_0\left(Y(t)\right)\varphi_0''\left(Y(t)\right)}{\left(\varphi_0'\left(Y(t)\right)\right)^2} - 1\right), \quad h(t) = \int\limits_{t_1}^t \frac{\sqrt{|H(\tau)|}}{\pi_\omega(\tau)}\,d\tau \quad (t_1 \in [a, \omega[),$$

$$\mu_2 = \text{sign } \varphi_0(y) \quad \text{for} \quad y \in \Delta_{Y_0}.$$

Theorem 3.1. *Let $\lambda_0 \in \mathbb{R} \setminus \{0,1\}$ and let the function f satisfies condition $(FN)_{\lambda_0}$. Then, for the existence of $P_\omega(Y_0, Y_1, \lambda_0)$- solutions of the differential equation (1.1), it is necessary that the conditions (1.2), (1.6) and*

$$\alpha_0\mu_0\lambda_0 > 0, \quad \mu_0\mu_1\lambda_0(\lambda_0-1)\pi_\omega(t) > 0, \quad \alpha_0\mu_2(\lambda_0-1)I(t) < 0 \quad \textit{for} \quad t \in [a,\omega[\tag{3.1}$$

$$\alpha_0(\lambda_0 - 1)\lim_{t\uparrow\omega} I(t) = Z_0, \tag{3.2}$$

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)I'(t)}{I(t)} = \pm\infty, \quad \lim_{t\uparrow\omega} \frac{\alpha_0(\lambda_0 - 1)\pi_\omega^2(t)p(t)\varphi_0(Y(t))}{Y(t)} = \frac{\lambda_0}{\lambda_0 - 1} \tag{3.3}$$

are hold.

Moreover, each solution of this kind admits the asymptotic representations

$$\frac{y'(t)}{\varphi_0(y(t))} = \alpha_0(\lambda_0-1)\pi_\omega(t)p(t)[1+o(1)], \quad \varphi_0'(y(t)) = -\frac{\lambda_0(1+o(1))}{(\lambda_0 - 1)I(t)} \quad \textit{as} \quad t \uparrow \omega. \tag{3.4}$$

Remark. Asymptotic representations of $P_\omega(Y_0, Y_1, \lambda_0)$- solutions of Eq. (1.1) can be written explicitly

$$y(t) = Y(t)\left(1 + \frac{o(1)}{H(t)}\right), \quad y'(t) = \frac{\lambda_0}{\lambda_0 - 1}\frac{Y(t)}{\pi_\omega(t)}(1 + o(1)) \quad \text{as} \quad t \uparrow \omega. \tag{3.5}$$

Proof of Theorem 3.1. Let $\lambda_0 \in \mathbf{R} \setminus \{0,1\}$ and $y : [t_0, \omega[\to \Delta_{Y_0}$ be an arbitrary $P_\omega(Y_0, Y_1, \lambda_0)-$ solution of Eq. (1.1). Then there is a number $t_1 \in [t_0, \omega[$ such that $y^{(k)}(t) \neq 0$ $(k = 0,1,2)$, $\operatorname{sign} y^{(k)}(t) = \mu_k$ $(k = 0,1)$ for $t \in [t_1, \omega[$. Moreover, from the equality

$$\left(\frac{y(t)}{y'(t)}\right)' = 1 - \frac{y(t)y''(t)}{(y'(t))^2}$$

and conditions (1.3), the definition of the $P_\omega(Y_0, Y_1, \lambda_0)-$ solution immediately implies that

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)y'(t)}{y(t)} = \frac{\lambda_0}{\lambda_0 - 1}, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)y''(t)}{y'(t)} = \frac{1}{\lambda_0 - 1}. \tag{3.6}$$

From this, in particular, it follows that the second of the sign representations (3.1) holds. The first of conditions (3.1) can be obtained from the definition of the $P_\omega(Y_0, Y_1, \lambda_0)-$ solution of Eq.(1.1). Due to (3.6) and the condition $(FN)_{\lambda_0}$ which the function f satisfies from (1.1) we have

$$y''(t) = \alpha_0 p(t)\varphi_0(y(t))[1 + o(1)] \quad \text{as} \quad t \uparrow \omega \tag{3.7}$$

or

$$\frac{y''(t)}{\varphi_0(y(t))} = \alpha_0 p(t)[1 + o(1)] \quad \text{as} \quad t \uparrow \omega. \tag{3.8}$$

Multiplying both parts of (3.8) by $\pi_\omega(t)$, taking into account (3.6), we obtain the first of relations (3.4), whose integration on the interval from t_1 to t leads to the limiting equality

$$\int_{t_1}^{t} \frac{y'(\tau)\, d\tau}{\varphi_0(y(\tau))} = \alpha_0(\lambda_0 - 1) \int_{t_1}^{t} \pi_\omega(\tau)p(\tau)\, d\tau\, [1 + o(1)] \quad \text{as} \quad t \uparrow \omega$$

or by virtue of the definition of the limits of integration A and B

$$\Phi(y(t)) = \alpha_0(\lambda_0 - 1)I(t)[1 + o(1)] \quad \text{as} \quad t \uparrow \omega. \tag{3.9}$$

It follows from condition (1.4) that the function φ_0 together with its derivative of the first order are rapidly varying as $y \to Y_o$, because $\lim\limits_{\substack{y \longrightarrow Y_o \\ y \in \Delta_{Y_0}}} \dfrac{y\varphi_0'(y)}{\varphi_0(y)} = \pm\infty$, $\quad \lim\limits_{\substack{y \longrightarrow Y_o \\ y \in \Delta_{Y_0}}} \dfrac{y\varphi_0''(y)}{\varphi_0'(y)} = \pm\infty.$

Also from (1.4) as $y \to Y_o$ the equivalence $\dfrac{\varphi_0'(y)}{\varphi_0(y)} \sim \dfrac{\varphi_0''(y)}{\varphi_0'(y)}$ follows. In addition, taking to account the L'Hopital rule in the form of Stolz, we can assert that

$$\lim_{\substack{y \longrightarrow Y_o \\ y \in \Delta_{Y_0}}} \frac{\Phi(y)}{\dfrac{1}{\varphi_0'(y)}} = \lim_{\substack{y \longrightarrow Y_o \\ y \in \Delta_{Y_0}}} \frac{\dfrac{1}{\varphi_0(y)}}{-\dfrac{\varphi_0''(y)}{(\varphi_0'(y))^2}} = -\lim_{\substack{y \longrightarrow Y_o \\ y \in \Delta_{Y_0}}} \frac{(\varphi_0(y))^2}{\varphi_0''(y)\varphi_0(y)}, \tag{3.10}$$

hence

$$\Phi(y) \sim -\frac{1}{\varphi_0'(y)} \quad \text{as} \quad y \to Y_o, \qquad \Phi(y)\varphi_0'(y) < 0 \quad \text{for} \quad y \in \Delta_{Y_0}. \tag{3.11}$$

Condition (3.11) implies as $y \to Y_o$ fulfillment of the equivalences

$$\frac{\Phi'(y)}{\Phi(y)} = \frac{\dfrac{1}{\varphi_0(y)}}{\Phi(y)} \sim -\frac{\varphi_0'(y)}{\varphi_0(y)}, \quad \frac{\Phi''(y)\Phi(y)}{(\Phi'(y))^2} = \frac{-\dfrac{\varphi_0'(y)}{\varphi_0^2(y)}\Phi(y)}{\dfrac{1}{\varphi_0^2(y)}} \sim 1. \tag{3.12}$$

Hence taking into account the lemma 2.14 (see [4], Chap. II, Sec. 2.3., P. 54) it follows that the functionΦ belongs to the class $\Gamma(Y_0, Z_0)$ with a complementary function g, for which one can choose one of the equivalent functions

$$\frac{\Phi'(y)}{\Phi''(y)} \sim \frac{\Phi(y)}{\Phi'(y)} \sim -\frac{\varphi_0(y)}{\varphi_0'(y)} \quad \text{as} \quad y \to Y_o.$$

Further from (3.9) by virtue of (3.11) the third of the sign conditions (3.1) follows. Next from (3.9), (3.11) we have

$$\frac{y'(t)\varphi_0'(y(t))}{\varphi_0(y(t))} = -\frac{\pi_\omega(t)p(t)}{I(t)}[1+o(1)],$$

which, by virtue (3.6), implies the equality

$$\frac{y(t)\varphi_0'(y(t))}{\varphi_0(y(t))} = -\frac{\lambda_0 - 1}{\lambda_0}\frac{\pi_\omega^2(t)p(t)}{I(t)}[1+o(1)] \quad \text{as} \quad t \uparrow \omega.$$

From here, with allowance, $\dfrac{y\varphi_0'(y)}{\varphi_0(y)} = \pm\infty$ the first of the limit relations (3.3) follows.

Note that the function Φ retains its sign on Δ_{Y_0}, tends either to 0 or to $\pm\infty$ as $y \to Y_0$, and increases on Δ_{Y_0} due to $\Phi'(y) > 0$. Therefore, it has an inverse function $\Phi^{-1} : \Delta_{Z_0} \to \Delta_{Y_0}$, where, due to the second of conditions (1.4) and the increase Φ^{-1}

$$Z_0 = \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \Phi(y) \in \{0, \pm\infty\}, \tag{3.13}$$

$$\Delta_{Z_0} = \begin{cases} [z_0, Z_0[& \text{if} \quad \Delta_{Y_0} \quad \text{is a left neighborhood of} \quad Y_0, \\ ]Z_0, z_0] & \text{if} \quad \Delta_{Y_0} \quad \text{is a right neighborhood of} \quad Y_0, \end{cases} \qquad z_0 = \Phi(b).$$

Now from (3.9) implies (3.2) and we find that

$$y(t) = \Phi^{-1}\left(\alpha_0(\lambda_0 - 1)I(t)[1+o(1)]\right) \quad \text{as} \quad t \uparrow \omega. \tag{3.14}$$

Note that the function Φ^{-1} belongs to the class $\Gamma(Y_0, Z_0)$ and complementary to it can be chosen as $g(y) = -\dfrac{\varphi_0(y)}{\varphi_0'(y)}$. From the definition of Z_0, μ_2, the third of sign conditions (3.1) $\Phi^{-1}\left(\alpha_0(\lambda_0 - 1)I(t)\right) \in \Delta_{Y_0}$ as $t \in [t_0, \omega[$ and $\lim\limits_{t\uparrow\omega} \Phi^{-1}\left(\alpha_0(\lambda_0 - 1)I(t)\right) = Y_0$ follow. Therefore, based on the lemma 2.3 we have the limit equality

$$\lim_{t\uparrow\omega}\frac{\Phi^{-1}\left(\alpha_0(\lambda_0-1)I(t)[1+o(1)]\right)-\Phi^{-1}\left(\alpha_0(\lambda_0-1)I(t)\right)}{-\dfrac{\varphi_0\left(\Phi^{-1}\left(\alpha_0(\lambda_0-1)I(t)\right)\right)}{\varphi_0'\left(\Phi^{-1}\left(\alpha_0(\lambda_0-1)I(t)\right)\right)}}=$$

$$=\lim_{\substack{z\to Z\\ z\in\Delta_Z}}\frac{\Phi^{-1}\left(z[1+o(1)]\right)-\Phi^{-1}\left(z\right)}{-\dfrac{\varphi_0\left(z\right)}{\varphi_0'\left(z\right)}}=0,$$

which we can rewrite in the form

$$\Phi^{-1}\left(\alpha_0(\lambda_0-1)I(t)[1+o(1)]\right)=$$
$$\Phi^{-1}\left(\alpha_0(\lambda_0-1)I(t)\right)+\frac{\varphi_0\left(\Phi^{-1}\left(\alpha_0(\lambda_0-1)I(t)\right)\right)}{\varphi_0'\left(\Phi^{-1}\left(\alpha_0(\lambda_0-1)I(t)\right)\right)}o(1)\quad\text{as}\quad t\uparrow\omega.$$

Thus, the first of (3.5) is established. Note that this relation can also be rewritten as

$$y(t)=Y(t)[1+o(1)]\quad\text{at}\quad t\uparrow\omega,\tag{3.15}$$

since

$$\lim_{t\uparrow\omega}\frac{Y(t)\varphi_0'\left(Y(t)\right)}{\varphi_0\left(Y(t)\right)}=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{y\varphi_0'(y)}{\varphi_0(y)}=\pm\infty.\tag{3.16}$$

Invoking the first of the limiting equalities (3.6), from (3.15) we obtain the second of the relations (3.5). Now we write (3.7) with using (3.5) in the form

$$y''(t)=\alpha_0p(t)\varphi_0\left(Y(t)+\frac{\varphi_0\left(Y(t)\right)}{\varphi_0'\left(Y(t)\right)}\right)[1+o(1)]\quad\text{as}\quad t\uparrow\omega.\tag{3.17}$$

Then, as a complementary to the function $\varphi_0\in\Gamma(Y_0,Z_0)$, we choose $g(y)=\frac{\varphi_0(y)}{\varphi_0'(y)}$. Then, taking into account that $\lim\limits_{t\uparrow\omega}Y(t)=Y_o$, $Y(t)\in\Delta_{Y_0}$ at $t\in[t_0,\omega]$, we obtain

$$\lim_{t\uparrow\omega}\frac{\varphi_0\left(Y(t)+\dfrac{\varphi_0(Y(t))}{\varphi_0'(Y(t))}o(1)\right)}{\varphi_0\left(Y(t)\right)}=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{\varphi_0\left(y+\dfrac{\varphi_0(y)}{\varphi_0'(y)}o(1)\right)}{\varphi_0(y)}=1,$$

which in turn leads to

$$\varphi_0\left(Y(t)+\frac{\varphi_0(Y(t))}{\varphi_0'(Y(t))}o(1)\right)=\varphi_0\left(Y(t)\right)[1+o(1)]\quad\text{as}\quad t\uparrow\omega.$$

Therefore, relation (3.17) takes the form

$$y''(t)=\alpha_0p(t)\varphi_0\left(Y(t)\right)[1+o(1)]\quad\text{as}\quad t\uparrow\omega.$$

From the last representation, taking into account the second of the limit equalities (3.6), we obtain he second of conditions (3.3).

The theorem and remark are proved.

To prove sufficient conditions for the existence $P_\omega(Y_0, Y_1, \lambda_0)-$solutions to Eq. (1.1) we need several auxiliary assertions.

Lemma 3.1. *1) If there is a (finite or equal $\pm\infty$) limit*

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\left(\dfrac{\varphi_0'(y)}{\varphi_0(y)}\right)'}{\left(\dfrac{\varphi_0'(y)}{\varphi_0(y)}\right)^2}\sqrt{\left|\frac{y\varphi_0'(y)}{\varphi_0(y)}\right|} \tag{3.18}$$

then it is equal to 0;

2) if there are (finite or equal $\pm\infty$) limits

$$\gamma = \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{y\left(\dfrac{\varphi_0'(y)}{\varphi_0(y)}\right)'}{\dfrac{\varphi_0'(y)}{\varphi_0(y)}}, \tag{3.19}$$

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{y\left(\dfrac{\varphi_0'(y)}{\varphi_0(y)}\right)'}{\dfrac{\varphi_0'(y)}{\varphi_0(y)}} \frac{\displaystyle\int_{y_0}^{y}\sqrt{\left|\frac{z\varphi_0'(z)}{\varphi_0(z)}\right|}\,dz}{\sqrt{\left|\dfrac{y\varphi_0'(y)}{\varphi_0(y)}\right|}} \quad \textit{for} \quad \gamma = \pm\infty, \tag{3.20}$$

then

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\left(\dfrac{\varphi_0'(y)}{\varphi_0(y)}\right)'}{\left(\dfrac{\varphi_0'(y)}{\varphi_0(y)}\right)^2}\sqrt{\left|\frac{y\varphi_0'(y)}{\varphi_0(y)}\right|} = 0 \tag{3.21}$$

and

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{y\left(\dfrac{\varphi_0'(y)}{\varphi_0(y)}\right)'}{\dfrac{\varphi_0'(y)}{\varphi_0(y)}} \frac{\displaystyle\int_{y_0}^{y}\sqrt{\left|\frac{z\varphi_0'(z)}{\varphi_0(z)}\right|}\,dz}{\sqrt{\left|\dfrac{y\varphi_0'(y)}{\varphi_0(y)}\right|}} = 2 \quad \textit{for} \quad \gamma = \pm\infty. \tag{3.22}$$

Theorem 3.2. *Let $\lambda_0 \in \mathbb{R}\setminus\{0,1\}$ and let the function f satisfies condition $(FN)_{\lambda_0}$ and conditions (3.1) - (3.3) are satisfied, as well as*

$$\lim_{t\uparrow\omega}\left(\frac{\lambda_0}{\lambda_0-1} - q(t)\right)\sqrt{|H(t)|} = 0. \tag{3.23}$$

Then:

1) if $\alpha_0\mu_2 = 1$ and exists (finite or equal to $\pm\infty$) limit (3.18), then differential equation (1.1) has a one-parameter family of $P_\omega(Y_0, Y_1, \lambda_0)-$solutions with asymptotic representations

$$y(t) = Y(t)\left(1 + \frac{o(1)}{H(t)}\right), \quad y'(t) = \frac{\lambda_0}{\lambda_0 - 1}\frac{Y(t)}{\pi_\omega(t)}\left(1 + \frac{o(1)}{\sqrt{|H(t)|}}\right); \tag{3.24}$$

2) if $\alpha_0\mu_2 = -1$ and there are (finite or equal to $\pm\infty$) limits (3.19), (3.20), then

$$\lim_{t\uparrow\omega}\left(\frac{\lambda_0}{\lambda_0 - 1} - q(t)\right)\sqrt{|H(t)|}h^2(t) = 0 \tag{3.25}$$

and in each of the cases: $|\gamma| < +\infty$, $\gamma \neq -1$, $3\lambda_0 - 2 + 5\lambda_0\gamma \neq 0$ or $\gamma = \pm\infty$ the differential equation (1.1) has at least one $P_\omega(Y_0, Y_1, \lambda_0)-$ solution with asymptotic representations

$$y(t) = Y(t)\left(1 + \frac{o(1)}{h(t)H(t)}\right), \quad y'(t) = \frac{\lambda_0}{\lambda_0 - 1}\frac{Y(t)}{\pi_\omega(t)}\left(1 + \frac{o(1)}{h(t)\sqrt{|H(t)|}}\right), \tag{3.26}$$

moreover, in the case $|\gamma| < +\infty$ and $\lambda_0(\gamma+1)(3\lambda_0 - 2 + 5\lambda_0\gamma) < 0$ there exists a two-parameter family of solutions with the indicated asymptotics.

Proof of Theorem 3.2. Let $\lambda_0 \in \mathbf{R} \setminus \{0, 1\}$, the function f satisfies condition $(FN)_{\lambda_0}$ and the conditions (3.1) - (3.3) hold, as well as (3.23). Let us show that for Eq.1 (1.1) there is at least one $P_\omega(Y_0, Y_1, \lambda_0)-$ solution in each of the cases: $\alpha_0\mu_2 = -1$, $\alpha_0\mu_2 = 1$.

Let's make a transformation

$$y(t) = Y(t)\left(1 + \frac{v_1}{H(t)}\right), \quad y'(t) = \frac{\lambda_0}{\lambda_0 - 1}\frac{Y(t)}{\pi_\omega(t)}\left(1 + \frac{v_2}{\sqrt{|H(t)|}}\right) \tag{3.27}$$

and obtain a system of differential equations of the form:

$$
\begin{cases}
v_1' &= \dfrac{\sqrt{|H(t)|}}{\pi_\omega(t)}\left[\left(\dfrac{\lambda_0}{\lambda_0-1}-q(t)\right)\sqrt{|H(t)|}\operatorname{sign} H(t)+\sqrt{|H(t)|}\operatorname{sign} H(t)k(t)v_1+\right.\\
&+\left.\dfrac{\lambda_0\operatorname{sign} H(t)}{\lambda_0-1}v_2\right],\\
\\
v_2' &= \dfrac{\sqrt{|H(t)|}}{\pi_\omega(t)}\left[\dfrac{f\left(t,Y(t,v_1),Y^{[1]}(t,v_2)\right)}{\alpha_0 p(t)\varphi_0\left(Y(t,v_1)\right)}\dfrac{q(t)}{\lambda_0}\dfrac{\varphi_0\left(Y(t,v_1)\right)}{\varphi_0\left(Y(t)\right)}+1-q(t)+\right.\\
&+\left.\left(1-\dfrac{q(t)}{2}+\dfrac{H(t)k(t)}{2}\right)\dfrac{v_2}{\sqrt{|H(t)|}}\right],
\end{cases}
\tag{3.28}
$$

where $Y(t,v_1)=Y(t)\left(1+\dfrac{v_1}{H(t)}\right)$, $Y^{[1]}(t,v_2)=\dfrac{\lambda_0}{\lambda_0-1}\dfrac{Y(t)}{\pi_\omega(t)}\left(1+\dfrac{v_2}{\sqrt{|H(t)|}}\right)$.

Consider system (3.28) on the set $D_0=[t_0,\omega[\times V_0$, where $V_0=\big\{(v_1,v_2):|v_i|\le\frac12$, $i=1,2\big\}$, and the number t_0 (taking into account (3.1), (3.2)) is chosen in such a way that $Y(t,v_1)\in\Delta_{Y_0}(b)$ for $t\in[t_0,\omega[$ and $|v_1|\le\frac12$; $Y^{[1]}(t,v_2)\in\Delta_{Y_1}$ at $t\in[t_0,\omega[$ and $|v_2|\le\frac12$; $\alpha_0(\lambda_0-1)I(t)\in\Delta_{Z_0}$ at $t\in[t_0,\omega[$.

By virtue of the limit equality (1.4), we have

$$\lim_{t\uparrow\omega}H(t)=\pm\infty. \tag{3.29}$$

Moreover, the second of conditions (3.3) implies

$$\lim_{t\uparrow\omega}q(t)=\frac{\lambda_0}{\lambda_0-1}. \tag{3.30}$$

A consequence of condition (3.23) is the limit equaty

$$\lim_{t\uparrow\omega}k(t)\sqrt{|H(t)|}=\lim_{t\uparrow\omega}\left(\frac{\lambda_0}{\lambda_0-1}-q(t)\right)\sqrt{|H(t)|}=0.$$

Then, taking into account (1.4), (3.29), (3.30)

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)\left(Y(t,v_1)\right)'_t}{Y(t,v_1)}=\lim_{t\uparrow\omega}q(t)+\lim_{t\uparrow\omega}\frac{\pi_\omega(t)H'(t)v_1}{H^2(t)\left(1+\dfrac{v_1}{H(t)}\right)}=$$

$$=\lim_{t\uparrow\omega}q(t)\left(1-\left(\frac{1}{H(t)}+\frac{\varphi_0''\left(Y(t)\right)\varphi_0\left(Y(t)\right)}{\left(\varphi_0'\left(Y(t)\right)\right)^2}-1\right)\frac{v_1}{1+\dfrac{v_1}{H(t)}}\right)=\frac{\lambda_0}{\lambda_0-1}$$

unformly in $|v_1|\le\frac12$.

Also, in view of (1.4), (3.29), (3.23), we obtain

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)\left(Y^{[1]}(t,v_2)\right)'_t}{Y^{[1]}(t,v_2)}=\lim_{t\uparrow\omega}q(t)-1-\frac{v_2}{2}\lim_{t\uparrow\omega}\frac{\pi_\omega(t)H'(t)}{H(t)\sqrt{|H(t)|}}=\frac{1}{\lambda_0-1}+$$
$$+\lim_{t\uparrow\omega}\frac{v_2}{2\left(1+\frac{v_2}{\sqrt{|H(t)|}}\right)}\left(\frac{q(t)}{\sqrt{|H(t)|}}+\operatorname{sign}H(t)\sqrt{|H(t)|}k(t)\right)=\frac{1}{\lambda_0-1}$$

unformly in $|v_2|\le\frac{1}{2}$.

Also, due to Definition 1.2

$$\lim_{t\uparrow\omega}\frac{f\left(t,Y(t,v_1),Y^{[1]}(t,v_2)\right)}{\alpha_0p(t)\varphi_0\left(Y(t,v_1)\right)}=1\quad\text{unformly in}\quad(v_1,v_2)\in V_0,$$

those we can write

$$\frac{f\left(t,Y(t,v_1),Y^{[1]}(t,v_2)\right)}{\alpha_0p(t)\varphi_0\left(Y(t,v_1)\right)}=1+R_1(t,v_1,v_2),\tag{3.31}$$

where $\lim\limits_{t\uparrow\omega}R_1(t,v_1,v_2)=0$ unformly in $(v_1,v_2)\in V_0$.

Next, consider the relationship $\dfrac{\varphi_0\left(Y(t,v_1)\right)}{\varphi_0\left(Y(t)\right)}$.

By Lemma 2.2

$\lim\limits_{t\uparrow\omega}\dfrac{\varphi_0\left(Y(t,v_1)\right)}{\varphi_0\left(Y(t)\right)}=\lim\limits_{t\uparrow\omega}\dfrac{\varphi_0\left(Y(t)+\frac{\varphi_0(Y(t))v_1}{\varphi_0'(Y(t))}\right)}{\varphi_0\left(Y(t)\right)}=e^{v_1}$, that's why $\dfrac{\varphi_0\left(Y(t,v_1)\right)}{\varphi_0\left(Y(t)\right)}=$ $1+v_1+R(t,v_1)$,

where

$R(t,v_1)=\dfrac{\varphi_0\left(Y(t,v_1)\right)}{\varphi_0\left(Y(t)\right)}-1-v_1$, $\lim\limits_{t\uparrow\omega}R(t,v_1)=0$ unformly in $|v_1|\le\frac{1}{2}$.

Moreover, it is easy to see that $R'_{v_1}(t,v_1)=\dfrac{\varphi_0'\left(Y(t)+\frac{\varphi_0(Y(t))v_1}{\varphi_0'(Y(t))}\right)}{\varphi_0'\left(Y(t)\right)}-1.$

Here $\varphi_0'\in\Gamma(Y_0,Z_0)$ with complementary function $g(y)=\dfrac{\varphi_0(y)}{\varphi_0'(y)}$. Hence, by Lemma 2.2

$\lim\limits_{t\uparrow\omega}R'_{v_1}(t,v_1)=e^{v_1}-1$ or $R'_{v_1}(t,v_1)=v_1+r(t,v_1)$, $\lim\limits_{t\uparrow\omega}r(t,v_1)=0$ unformly in

$|v_1|\le\frac{1}{2}$. Hence, for any $\varepsilon>0$ there also exist $t_1\in[t_0,\omega[$ and $\delta\in]0,\frac{1}{2}]$ such that $|R'_{v_1}(t,v_1)|\le\varepsilon$ at $t\in[t_1,\omega[$, $v_1\in[-\delta,\delta]$.

Therefore, the function R satisfies the Lipschitz condition in terms of the variable v_1 with the Lipschitz constantε, we have the estimate (taking into account $R(t,0)=0$)

$$|R(t,v_1)| \le \varepsilon |v_1| \quad \text{at} \quad t \in [t_1,\omega[\quad \text{and} \quad v_1 \in [-\delta,\delta].$$

Moreover, if we use the Maclaurin formula with a remainder term in the Lagrange form up to a term of the second order, we can write

$$R(t,v_1) = \frac{1}{2}\frac{\varphi_0(Y(t))}{(\varphi_0'(Y(t))^2}\varphi_0''\left(Y(t)+\frac{\varphi_0(Y(t))}{\varphi_0'(Y(t))}\xi\right)v_1^2, \text{ where } |\xi| \le |v_1|.$$

In view of (1.4)

$$\varphi_0''\left(Y(t)+\frac{\varphi_0(Y(t))}{\varphi_0'(Y(t))}\xi\right) = \frac{\left(\varphi_0'\left(Y(t)+\dfrac{\varphi_0(Y(t))}{\varphi_0'(Y(t))}\xi\right)\right)^2}{\varphi_0\left(Y(t)+\dfrac{\varphi_0(Y(t))}{\varphi_0'(Y(t))}\xi\right)}(1+r_1(t,v_1)),$$

where

$\lim\limits_{t\uparrow\omega} r_1(t,v_1)=0$ unformly in $|v_1| \le \frac{1}{2}$. Further, taking into account that $\varphi_0, \varphi_0' \in \Gamma(Y_0,Z_0)$ have the complementary function $g(y) = \dfrac{\varphi_0(y)}{\varphi_0'(y)}$, based on Lemma 2.2, we have

$\varphi_0''\left(Y(t)+\dfrac{\varphi_0(Y(t))}{\varphi_0'(Y(t))}\xi\right) = \dfrac{(\varphi_0'(Y(t)))^2}{\varphi_0(Y(t))}e^{\xi}(1+r_2(t,v_1))$, where $\lim\limits_{t\uparrow\omega} r_2(t,v_1) =$ 0

unformly in $|v_1| \le \frac{1}{2}$.

Therefore

$$R(t,v_1) = \frac{1}{2}e^{\xi}(1+r_1(t,v_1))(1+r_2(t,v_1))v_1^2. \tag{3.32}$$

Hence, for any $\varepsilon > 0$ there also exist $t_1 \in [t_0,\omega[$ and $\delta \in]0,\frac{1}{2}]$ such that

$$|R(t,v_1)| \le (1+\varepsilon)|v_1^2| \quad \text{as} \quad t \in [t_1,\omega[, \quad v_1 \in [-\delta,\delta]. \tag{3.33}$$

Having chosen an arbitrary ε system (3.28) we will further consider it on the set $D_1 = [t_1,\omega[\times V_1$, where $V_1 = \{(v_1,v_2) : |v_i| \le \delta, \quad i=1,2\}$.

We now rewrite system (3.28) in the form

$$\begin{cases} v_1' = \dfrac{\sqrt{|H(t)|}}{\pi_\omega(t)}\left[f_1(t)+c_{11}(t)v_1+c_{12}(t)v_2\right], \\ \\ v_2' = \dfrac{\sqrt{|H(t)|}}{\pi_\omega(t)}\left[f_2(t)+c_{21}(t)v_1+c_{22}(t)v_2+V_1(t,v_1,v_2)+V_2(t,v_1,v_2)\right], \end{cases} \tag{3.34}$$

where

$$f_1(t)=\Big(\frac{\lambda_0}{\lambda_0-1}-q(t)\Big)\sqrt{|H(t)|}\,\mathrm{sign}\,H(t),\quad c_{11}(t)=\sqrt{|H(t)|}\,\mathrm{sign}\,H(t)k(t),$$
$$c_{12}(t)=\frac{\lambda_0\,\mathrm{sign}\,H(t)}{\lambda_0-1},\quad f_2(t)=\frac{q(t)}{\lambda_0}+1-q(t),\quad c_{21}(t)=\frac{q(t)}{\lambda_0};$$
$$c_{22}(t)=\left(1-\frac{q(t)}{2}+\frac{H(t)k(t)}{2}\right)\frac{1}{\sqrt{|H(t)|}},V_1(t,v_1,v_2)=\frac{q(t)}{\lambda_0}R_1(t,v_1,v_2)(1+v_1),$$
$$V_2(t,v_1,v_2)=\frac{q(t)}{2\lambda_0}\left(1+R_1(t,v_1,v_2)\right)e^{\xi}\left(1+r_1(t,v_1)\right)\left(1+r_2(t,v_1)\right)v_1^2.$$

Here, in view of (3.23), (3.29), (3.30), (3.18), the representation of the function R_1, we have

$$\lim_{t\uparrow\omega}f_1(t)=0,\quad \lim_{t\uparrow\omega}f_2(t)=0,\quad \lim_{t\uparrow\omega}c_{11}(t)=0,\quad \lim_{t\uparrow\omega}c_{12}(t)=\frac{\lambda_0\mu_0\mu_2}{\lambda_0-1},\quad \lim_{t\uparrow\omega}c_{21}(t)=$$
$$=\frac{1}{\lambda_0-1},\quad \lim_{t\uparrow\omega}h(t)=\pm\infty,\quad \lim_{t\uparrow\omega}V_1(t,v_1,v_2)=0\quad \text{unformly in}\quad (v_1,v_2)\in V_1,$$
$$\lim_{|v_1|+|v_2|\to 0}\frac{V_2(t,v_1,v_2)}{|v_1|+|v_2|}=0\quad \text{unformly in}\quad t\in[t_1,\omega[. \tag{3.35}$$

In this case, the limit matrix of coefficients for v_1, v_2 has the form

$$\begin{pmatrix} 0 & \dfrac{\lambda_0\mu_0\mu_2}{\lambda_0-1}\\ \dfrac{1}{\lambda_0-1} & 0\end{pmatrix},$$

whose characteristic equation

$$\rho^2-\frac{\lambda_0\mu_0\mu_2}{(\lambda_0-1)^2}=0. \tag{3.36}$$

Note that $\mathrm{sign}(\mu_0\mu_2\lambda_0)=\alpha_0\mu_2\in\{-1,1\}$. If we assume that $\alpha_0\mu_2=1$ and (3.18) exists then Eq. (3.36) has two different roots of different signs and, therefore, according to Theorem 2.2 in [4], system (3.34) has a one-parameter family of solutions $(v_1,v_2):[t_2,\omega[\to R^2$ ($t_2\in[t_1,\omega[$) vanishing at $t\uparrow\omega$. By virtue of the transformation (3.27), each such system solution $y:[t_2,\omega[\to R$ corresponds to a solution $y:[t_2,\omega[\to R$ of Eq. (1.1) that admits the asymptotic representations (3.24) for $t\uparrow\omega$.

Now consider the situation when $\alpha_0\mu_2=-1$. In this case, the characteristic equation (3.36) has two complex conjugate roots. In order to be able to use the theorem (2.2) of [4], we apply two transformations to the system (3.34) in succession. Let's set the first one like this:

$$v_1(t) = w_1(\tau), \quad v_2(t) = w_2(\tau) + \tfrac{c}{\tau} w_1(\tau), \quad \tau(t) = \beta h(t),$$

$$\beta = \operatorname{sign}(\pi_\omega(t)) = \begin{cases} 1 & \text{if} \quad \omega = +\infty, \\ -1 & \text{if} \quad \omega < +\infty, \end{cases} \tag{3.37}$$

where the constant C will be chosen later. Note that

$$\tau(t_1) = 0, \quad \tau'(t) > 0 \quad \text{at} \quad t \in]t_1, \omega[\quad \lim_{t \uparrow \omega} \tau(t) = +\infty$$

Transformation (3.37) will bring system (3.34) to the form

$$\begin{cases} w_1' & = \beta \left[g_1(\tau) + m_{11}(\tau) w_1 - \dfrac{|\lambda_0|}{\lambda_0 - 1} w_2 \right], \\ \\ w_2' & = \beta \left[g_2(\tau) + \dfrac{w_1}{\lambda_0 - 1} + m_{21}(\tau) w_1 + m_{22}(\tau) w_2 + W_1(\tau, w_1, w_2) + W_2(\tau, w_1, w_2) \right], \end{cases} \tag{3.39}$$

where

$$g_1(\tau(t)) = f_1(t), \quad m_{11}(\tau(t)) = c_{11}(t) - \frac{|\lambda_0|}{\lambda_0 - 1} \frac{C}{\tau(t)}, \qquad g_2(\tau(t)) = f_2(t) - \frac{C}{\tau(t)} f_1(t),$$

$$m_{21}(\tau(t)) = c_{21}(t) - \frac{1}{\lambda_0 - 1} + \frac{C}{\tau(t)}(c_{22}(t) - c_{11}(t)) + \frac{C\beta}{\tau^2(t)} + \frac{C^2 |\lambda_0|}{\tau^2(t)(\lambda_0 - 1)},$$

$$m_{22}(\tau(t)) = c_{22}(t) + \frac{C|\lambda_0|}{\tau(t)(\lambda_0 - 1)}, \quad W_i(\tau(t), w_1, w_2) = V_i(t, w_1, w_2 + \tfrac{c}{\tau} w_1) \quad (i = 1, 2).$$

Here according to (3.29), (3.30), (3.35)

$$\begin{gathered} \lim_{\tau \to +\infty} g_i(\tau) = 0, \quad \lim_{\tau \to +\infty} m_{ii}(\tau) = 0 \quad (i = 1, 2), \lim_{\tau \to +\infty} m_{21}(\tau) = 0, \\ \lim_{\tau \to +\infty} W_1(\tau, w_1, w_2) = 0 \quad \text{unformly in} \quad (w_1, w_2) \in V_1, \\ \lim_{|w_1| + |w_2| \to 0} \frac{W_2(\tau, w_1, w_2)}{|w_1| + |w_2|} = 0 \quad \text{unformly in} \quad \tau \in [0, +\infty[. \end{gathered} \tag{3.40}$$

For further evaluations, we need to know the behavior of some expressions. It is easy to see that for $\gamma = \text{const}$, the function $\sqrt{\left| \dfrac{\varphi_0'(y)}{y \varphi_0(y)} \right|}$ is regularly varying $\dfrac{\gamma - 1}{2}$ order for $y \to Y_0$, because

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{y \left(\sqrt{\left| \dfrac{\varphi_0'(y)}{y \varphi_0(y)} \right|} \right)'}{\sqrt{\left| \dfrac{\varphi_0'(y)}{y \varphi_0(y)} \right|}} = \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{y \left(\dfrac{\varphi_0'(y)}{y \varphi_0(y)} \right)'}{2 \dfrac{\varphi_0'(y)}{y \varphi_0(y)}} = \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\left(\dfrac{\varphi_0'(y)}{\varphi_0(y)} \right)' - \dfrac{\varphi_0'(y)}{y \varphi_0(y)}}{2 \dfrac{\varphi_0'(y)}{y \varphi_0(y)}} =$$

$$\frac{\gamma - 1}{2}.$$

Further, based on the form of the functions H, q, h, (3.29), (3.30), (3.3), it is easy to verify by L'Hospital's rule that

$$\lim_{t\uparrow\omega}\frac{h(t)}{\int\limits_{Y(t_1)}^{Y(t)}\sqrt{\left|\frac{z\varphi_0'(z)}{\varphi_0(z)}\right|}\,\frac{dz}{z}}=\lim_{t\uparrow\omega}\frac{\int\limits_{t_1}^{t}\sqrt{\left|\frac{Y(s)\varphi_0'(Y(s))}{\varphi_0(Y(s))}\right|}\,\frac{Y'(s)}{Y(s)q(s)}ds}{\int\limits_{Y(t_1)}^{Y(t)}\sqrt{\left|\frac{z\varphi_0'(z)}{\varphi_0(z)}\right|}\,\frac{dz}{z}}=\lim_{t\uparrow\omega}\frac{1}{q(t)}=\frac{\lambda_0-1}{\lambda_0}. \tag{3.41}$$

Using the representation of a regularly varying function and the properties of slowly varying functions, from the last limit equality at $t\uparrow\omega$ we have

$$h(t)\sim\frac{1}{q(t)}\int\limits_{Y(t_1)}^{Y(t)}\sqrt{\left|\frac{z\varphi_0'(z)}{\varphi_0(z)}\right|}\,\frac{dz}{z}\sim\frac{\lambda_0-1}{\lambda_0}\frac{2}{\gamma+1}\sqrt{|H(t)|}. \tag{3.42}$$

Now we can find

$$\lim_{\tau\to+\infty}\tau(m_{22}(\tau)+m_{11}(\tau))=\lim_{t\uparrow\omega}\beta h(\tau(t))(c_{22}(t)+c_{11}(t))=\lim_{t\uparrow\omega}\beta\frac{h(\tau(t))}{\sqrt{|H(\tau(t))|}}\times$$
$$\times\left(1-\frac{q(\tau(t))}{2}+\frac{3k(\tau(t))H(\tau(t))}{2}\right)=\frac{\beta(\lambda_0-2+3\lambda_0\gamma)}{\lambda_0(\gamma+1)}. \tag{3.43}$$

It is also possible to choose a constant C in such a way that $\tau(m_{22}(\tau)-m_{11}(\tau))$ as $\tau\to+\infty$ tends to zero. Really,

$$\lim_{\tau\to+\infty}\tau(m_{22}(\tau)-m_{11}(\tau))=\lim_{t\uparrow\omega}\beta h(\tau(t))\left(c_{22}(t)-c_{11}(t)+\frac{2|\lambda_0|C}{(\lambda_0-1)\tau(t)}\right)=$$
$$\lim_{t\uparrow\omega}\beta\frac{h(\tau(t))}{\sqrt{|H(\tau(t))|}}\left(1-\frac{q(\tau(t))}{2}-\frac{k(\tau(t))H(\tau(t))}{2}\right)+\frac{2|\lambda_0|C}{\lambda_0-1}=\frac{\beta(\lambda_0-\lambda_0\gamma-2)}{\lambda_0(\gamma+1)}$$
$$+\frac{2|\lambda_0|C}{\lambda_0-1}.$$

Thus, choosing at $\gamma\neq-1$

$$C=\frac{\beta(\lambda_0-1)(\lambda_0\gamma+2-\lambda_0)}{2|\lambda_0|\lambda_0(\gamma+1)} \tag{3.44}$$

we get

$$\lim_{\tau\to+\infty}\tau(m_{22}(\tau)-m_{11}(\tau))=0. \tag{3.45}$$

Also taking into account (3.38), (3,23), (3.42), (3.45)

$$\lim_{\tau\to+\infty}\tau m_{21}(\tau)=0. \tag{3.46}$$

If $\gamma = \pm\infty$ then due to (3.22)

$$\lim_{\tau\to+\infty} \tau(m_{22}(\tau)+m_{11}(\tau)) = \frac{3\beta}{2} \lim_{\tau\to+\infty} \frac{Y(t)\left(\dfrac{\varphi'(Y(t))}{\varphi(Y(t))}\right)}{\dfrac{\varphi'(Y(t))}{\varphi(Y(t))}} \frac{\int\limits_{Y(t_1)}^{Y(t)} \sqrt{\left|\dfrac{z\varphi_0'(z)}{\varphi_0(z)}\right|}\, \dfrac{dz}{z}}{\sqrt{\left|\dfrac{Y(t)\varphi_0'(Y(t))}{\varphi_0(Y(t))}\right|}} = 3\beta, \tag{3.47}$$

$$\lim_{\tau\to+\infty} \tau(m_{22}(\tau) - m_{11}(\tau)) = \frac{2|\lambda_0|C}{\lambda_0 - 1} - \beta$$

and the constant C can be chosen as

$$C = \frac{\beta(\lambda_0 - 1)}{2|\lambda_0|}, \tag{3.48}$$

(3.46) is hold at $\gamma = \pm\infty$. We now apply to system (3.39) the transformation

$$\begin{pmatrix} w_1(\tau) \\ \\ w_2(\tau) \end{pmatrix} = \begin{pmatrix} cos(\alpha\tau) & -sin(\alpha\tau) \\ \\ \dfrac{sin(\alpha\tau)}{\sqrt{|\lambda_0|}} & \dfrac{cos(\alpha\tau)}{\sqrt{|\lambda_0|}} \end{pmatrix} \begin{pmatrix} \dfrac{z_1(\tau)}{\tau} \\ \\ \dfrac{z_2(\tau)}{\tau} \end{pmatrix} \tag{3.49}$$

where $\alpha = \dfrac{\beta\sqrt{|\lambda_0|}}{\lambda_0 - 1}$. Let's get the system

$$\begin{cases} z_1' &= \frac{1}{\tau}\left[F_1(\tau) + b_{11}(\tau)z_1 + b_{12}(\tau)z_2 + Z_{11}(\tau, z_1, z_2) + Z_{12}(\tau, z_1, z_2)\right], \\ \\ z_2' &= \frac{1}{\tau}\left[F_2(\tau) + b_{21}(\tau)z_1 + b_{22}(\tau)z_2 + Z_{21}(\tau, z_1, z_2) + Z_{22}(\tau, z_1, z_2)\right], \end{cases} \tag{3.50}$$

where

$$
\begin{aligned}
&F_1(\tau)=\beta\tau^2\left(g_1(\tau)cos(\alpha\tau)+\sqrt{|\lambda_0|}g_2(\tau)sin(\alpha\tau)\right),\\
&F_2(\tau)=\beta\tau^2\left(-g_1(\tau)sin(\alpha\tau)+\sqrt{|\lambda_0|}g_2(\tau)cos(\alpha\tau)\right),\\
&b_{11}(\tau)=1+\tfrac{\beta\tau}{2}\left(m_{11}(\tau)+m_{22}(\tau)+(m_{11}(\tau)-m_{22}(\tau))cos(2\alpha\tau)+\right.\\
&\qquad\left.+\sqrt{|\lambda_0|}m_{21}(\tau)sin(2\alpha\tau)\right),\\
&b_{12}(\tau)=\tfrac{\beta\tau}{2}\left((m_{22}(\tau)-m_{11}(\tau))sin(2\alpha\tau)-2\sqrt{|\lambda_0|}m_{21}(\tau)sin^2(\alpha\tau)\right),\\
&b_{21}(\tau)=\tfrac{\beta\tau}{2}\left((m_{22}(\tau)-m_{11}(\tau))sin(2\alpha\tau)-2\sqrt{|\lambda_0|}m_{21}(\tau)cos^2(\alpha\tau)\right),\\
&b_{22}(\tau)=1+\tfrac{\beta\tau}{2}\left(m_{11}(\tau)+m_{22}(\tau)+(m_{22}(\tau)-m_{11}(\tau))cos(2\alpha\tau)-\right.\\
&\qquad\left.-\sqrt{|\lambda_0|}m_{21}(\tau)sin(2\alpha\tau)\right),\\
&Z_{1i}(\tau,z_1,z_2)=\beta\sqrt{|\lambda_0|}\tau^2sin(\alpha\tau)\times\\
&\times W_i(\tau,\tfrac{z_1}{\tau}cos(\alpha\tau)-\tfrac{z_2}{\tau}sin(\alpha\tau),\tfrac{z_1}{\sqrt{|\lambda_0|}\,\tau}sin(\alpha\tau)+\tfrac{z_2}{\sqrt{|\lambda_0|}\,\tau}cos(\alpha\tau))\quad(i=1,2),\\
&Z_{2i}(\tau,z_1,z_2)=\beta\sqrt{|\lambda_0|}\tau^2cos(\alpha\tau)\times\\
&\times W_i(\tau,\tfrac{z_1}{\tau}cos(\alpha\tau)-\tfrac{z_2}{\tau}sin(\alpha\tau),\tfrac{z_1}{\sqrt{|\lambda_0|}\,\tau}sin(\alpha\tau)+\tfrac{z_2}{\sqrt{|\lambda_0|}\,\tau}cos(\alpha\tau))\quad(i=1,2).
\end{aligned}
$$

Now, based on conditions (3.25), (3.43) - (3.48), we note that

$$\lim_{\tau\to+\infty}F_i(\tau)=0\ (i=1,2),\quad \lim_{\tau\to+\infty}b_{12}(\tau)=0,\quad \lim_{\tau\to+\infty}b_{21}(\tau)=0,$$

$$\lim_{\tau\to+\infty}b_{11}(\tau)=\lim_{\tau\to+\infty}b_{22}(\tau)=\begin{cases}\dfrac{3\lambda_0-2+5\lambda_0\gamma}{2\lambda_0(\gamma+1)} & \text{at}\quad \gamma=\text{const}\neq-1,\\[2ex] \dfrac{5}{2} & \text{at}\quad \gamma=\pm\infty.\end{cases}$$

In addition, according to (3.40), the boundedness of the functions sine, cosine

$$\lim_{|z_1|+|z_2|\to0}\frac{Z_{i2}(\tau,w_1,w_2)}{|z_1|+|z_2|}=0\quad\text{unformly in}\quad \tau\in[0,+\infty[\quad(i=1,2).\tag{3.51}$$

Moreover,

$$\lim_{\tau\to+\infty}Z_{1i}(\tau,z_1,z_2)=0\quad\text{unformly in}\quad(z_1,z_2)\in V_1\quad(i=1,2).\tag{3.52}$$

Then it follows from Theorem 2.2 in [4] that system (3.50) in each of the cases $|\gamma|<+\infty$, $\gamma\neq-1$, $3\lambda_0-2+5\lambda_0\gamma\neq0$ or $\gamma=\pm\infty$ has at least one solution $(z_1, z_2) : [\tau_*, +\infty[\to R^2$ ($\tau_* \in [0, +\infty[$) tending to zero as $\tau \to +\infty$. Also the inequality $\lambda_0(\gamma+1)(3\lambda_0 - 2 + 5\lambda_0\gamma) < 0$ is satisfied, there exists a two-parameter family of such solutions. According to the transformations (3.27), (3.34), (3.37), (3.49) each of these solutions corresponds to a solution of equation (1.1), for which the asymptotic representations (3.26) are valid for $t \uparrow \omega$.

The theorem is completely proven.

For instance, the differential equation $y'' = \sum\limits_{i=1}^{k} \alpha_{i0} p_i(t) \varphi_{i0}(y)$, where there is $j \in \{1, ..., k\}$ such that $p_j(t) = \dfrac{|c\lambda_0|}{(\lambda_0-1)^2} \dfrac{|\pi_\omega(t)|^{\frac{\lambda_0}{\lambda_0-1}-2}}{\varphi_{j0}\left(|\pi_\omega(t)|^{\frac{\lambda_0}{\lambda_0-1}}\right)}$, $\lim\limits_{t\uparrow\omega} \dfrac{p_i(t)}{p_j(t)} = 0$ $(j \in \{1, ..., k\}, i \neq j)$, $\lim\limits_{\substack{y \longrightarrow Y_o \\ y \in \Delta_{Y_0}}} \dfrac{\varphi_{i0}(y)}{\varphi_{j0}(y)} = 1$ $(j \in \{1, ..., k\})$, $c = const$, $c\mu_0 > 0$ satisfies the conditions of Theorem 1, Theorem 2 as $\lambda_0 \in \mathbf{R} \setminus \{0, 1\}$. Here $Y(t) = c|\pi_\omega(t)|^{\frac{\lambda_0}{\lambda_0-1}}$, $H(t) = \dfrac{Y(t)\varphi'_{j0}(Y(t))}{\varphi_{j0}(Y(t))}$.

Conclusion

In this paper, for essentially nonlinear nonautonomous differential equations of the second order in a sense, closed to two-term equations with rapidly varying nonlinearity with respect to the desired function, necessary and also sufficient conditions of existence and asymptotic representations of $P_\omega(Y_0, Y_1, \lambda_0)-$ solutions for $\lambda_0 \in \mathbf{R} \setminus \{0, 1\}$ at $t \uparrow \omega$ $(\omega \leq +\infty)$ are established. In the future, it will be of interest to obtain similar results in the cases $\lambda_0 = 1$, $\lambda_0 = 0$.

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Кусік Л. І.

Про асимптотичні зображення одного класу розв'язків диференціальних рівнянь другого порядку

Резюме

Для диференціального рівняння другого порядку загального виду $y'' = f(t, y, y')$, де $f : [a, \omega[\times \Delta_{Y_0} \times \Delta_{Y_1} \longrightarrow \mathbf{R}$ – неперервна функція, $-\infty < a < \omega \leq +\infty$, Δ_{Y_i} – односторонній окіл Y_i, $Y_i \in \{0, \pm\infty\}$ $(i \in \{0, 1\})$ розглянуто питання існування розв'язків, для яких $\lim\limits_{t\uparrow\omega} y^{(i)}(t) = Y_i$ $(i \in \{0, 1\})$. Серед множини таких розв'язків відокремлюємо достатньо широкий клас т. з. $P_\omega(Y_0, Y_1, \lambda_0)$-розв'язків. Такого типу розв'язки раніше було уведено при вивченні двочленного рівняння $y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y')$, де $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\longrightarrow]0, +\infty[$–неперервна функція, $\varphi_i : \Delta_{Y_i} \longrightarrow]0, +\infty[$ $(i = 0, 1)$ – неперервні правильно змінні при $z \to Y_i$ $(i = 0, 1)$ функції порядків σ_i $(i = 0, 1)$, $\sigma_0 + \sigma_1 \neq 1$. У даній роботі встановлено умову, за якій права частина рівняння в деякому сенсі є близькою при $\lambda_0 \in \mathbb{R}\backslash\{0, 1\}$ та $t \uparrow \omega$ до добутку $\alpha_0 p(t)\varphi_0(y)$, де функція φ_0 є швидко змінною при $y \to Y_0$. При виконанні цієї умови знайдено необхідні, а також достатні умови існування $P_\omega(Y_0, Y_1, \lambda_0)$-розв'язків, встановлено асимптотичні зображення таких розв'язків та їх похідних першого порядку, вказано кількість параметричних сімей таких розв'язків. Наведено приклад.

Ключові слова: двочленне диференціальне рівняння, $P_\omega(Y_0, Y_1, \lambda_0)$-розв'язок, асимптотичні зображення розв'язків, швидко змінна функція, одно-, двопараметрична сім'я розв'язків .

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