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NECESSARY EXISTENCE CONDITIONS OF $P_\omega(Y_0, Y_1, \lambda_0)$-SOLUTIONS OF SECOND-ORDER DIFFERENTIAL EQUATION WITH RAPIDLY VARYING NONLINIARITY

We consider a differential equation of the second order of the general form $y'' = f(t, y, y')$, where $f : [a, \omega[\times\Delta_{Y_0} \times \Delta_{Y_1} \longrightarrow \mathbf{R}$– a continuous function, $-\infty < a < \omega \le +\infty$, Δ_{Y_i}– one-side neighborhood of Y_i, $Y_i \in \{0, \pm\infty\}$ $(i \in \{0, 1\})$. Under certain conditions for the function f, this equation can be represented close to the two-term differential equation, namely $y'' = \alpha_0 p(t)\varphi_1(y')(1+o(1))$ at $t \uparrow \omega$, where φ_1 is a rapidly varying functionat $y' \to Y_1$. Found the necessary conditions for the existence of solutions for which $\lim\limits_{t\uparrow\omega} y^{(i)}(t) = Y_i$ $(i \in \{0, 1\})$, $\lim_{t\uparrow\omega} \dfrac{[y'(t)]^2}{y(t)y''(t)} = \lambda_0$, so called $P_\omega(Y_0, Y_1, \lambda_0)$-solutions. This type of solution was previously presented in works by Evtukhov V.M., Belozerova M.O. when studying the two-term equation $y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y')$, where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\longrightarrow]0, +\infty[$–continuous function, $\varphi_i : \Delta_{Y_i} \longrightarrow]0, +\infty[$ $(i = 0, 1)$ –continuous regularly variables for $z \to Y_i$ $(i = 0, 1)$ functions of orders σ_i $(i = 0, 1)$, and $\sigma_0 + \sigma_1 \ne 1$. Further, in the studies of V.M. Evtukhov, A.G. Chernikova for equation $y'' = \alpha_0 p(t)\varphi_0(y)$ necessary and sufficient conditions are established existence, as well as asymptotic at $t \uparrow \omega$ representations $P_\omega(Y_0, Y_1, \lambda_0)$-solutions in the case when φ_0 is a rapidly varying function at $y \to Y_0$.

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Introduction

Consider the differential equation

$$y'' = f(t, y, y'), \tag{1}$$

where $f : [a, \omega[\times\Delta_{Y_0} \times \Delta_{Y_1} \longrightarrow \mathbf{R}$ is continuous function, $-\infty < a < \omega \le +\infty$, Δ_{Y_i} $(i \in \{0, 1\})$ is a one-side neighborhood of Y_i and Y_i $(i \in \{0, 1\})$ is either 0

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or $\pm\infty$. We assume that the numbers μ_i $(i = 0, 1)$ given by the formula

$$\mu_i = \begin{cases} \;\;1 & \text{if eigher } Y_i = +\infty, \quad \text{or} \quad Y_i = 0 \\ & \text{and} \quad \Delta_{Y_i} \quad \text{is right neighborhood of the point } 0, \\ -1 & \text{if eigher } Y_i = -\infty, \quad \text{or} \quad Y_i = 0 \\ & \text{and} \quad \Delta_{Y_i} \quad \text{is left neighborhood of the point } 0, \end{cases}$$

satisfy the relations

$$\mu_0\mu_1 > 0 \quad \text{for} \quad Y_0 = \pm\infty \quad \text{and} \quad \mu_0\mu_1 < 0 \quad \text{for} \quad Y_0 = 0. \tag{2}$$

Conditions (2) are necessary for the existence of solutions of Eq. (1) defined in a left neighborhood of ω and satisfying the conditions

$$y^{(i)}(t) \in \Delta_{Y_i} \qquad \text{for } \; t \in [t_0, \omega[, \qquad \lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \quad (i = 0, 1). \tag{3}$$

One of the classes of Eq. (1) solutions with properties (3) that admits some asymptotic representations is the class of $P_\omega(Y_0, Y_1, \lambda_0)$- solutions.

Definition 1. *A solution y of Eq. (1) on interval $[t_0, \omega[\subset [a, \omega[$ is called $P_\omega(Y_0, Y_1, \lambda_0)$- solution, where $-\infty \le \lambda_0 \le +\infty$, if, in addition to (3), it satisfies the condition*

$$\lim_{t\uparrow\omega} \frac{[y'(t)]^2}{y(t)y''(t)} = \lambda_0.$$

Depending on λ_0 these solutions have different asymptotic properties. In [1] such ratios

$$\text{for} \quad \lambda_0 \in \mathbb{R} \setminus \{1\} \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)y'(t)}{y(t)} = \frac{\lambda_0}{\lambda_0 - 1}, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)y''(t)}{y'(t)} = \frac{1}{\lambda_0 - 1},$$

$$\text{for} \quad \lambda_0 = 1 \qquad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)y'(t)}{y(t)} = \pm\infty, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)y''(t)}{y'(t)} = \pm\infty,$$

$$\text{for} \quad \lambda_0 = \pm\infty \qquad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)y'(t)}{y(t)} = 1, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)y''(t)}{y'(t)} = 0,$$

where

$$\pi_\omega(t) = \begin{cases} t & \text{if} \quad \omega = +\infty, \\ t - \omega & \text{if} \quad \omega < +\infty, \end{cases}$$

are established.

Now consider a case $\lambda_0 \in \mathbb{R} \setminus \{1\}$. We impose a condition on the function f so that it becomes a two-term of a special form.

Definition 2. *We say that a function f satisfies condition $(FN1)_{\lambda_0}$ for $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$ if there exist a number $\alpha_0 \in \{-1, 1\}$, a continuous function*

$p : [a, \omega[\longrightarrow]0, +\infty[$ *and twice continuously differentiable function* $\varphi_1 : \Delta_{Y_1} \longrightarrow\,]0, +\infty[$, *satisfying the conditions*

$$\varphi_1'(w) \neq 0, \quad \lim_{\substack{w \longrightarrow Y_1 \\ w \in \Delta_{Y_1}}} \varphi_1(w) = \varphi_1 \in \{0, +\infty\}, \quad \lim_{\substack{w \longrightarrow Y_1 \\ w \in \Delta_{Y_1}}} \frac{\varphi_1(w)\varphi_1''(w)}{(\varphi_1'(w))^2} = 1, \quad (4)$$

such that, for arbitrary continuously differentiable functions $z_i : [a, \omega[\longrightarrow \Delta_{Y_i}$ $(i = 0, 1)$, *satisfying the conditions*

$$\lim_{t \uparrow \omega} z_i(t) = Y_i \quad (i = 0, 1),$$

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) z_0'(t)}{z_0(t)} = \frac{\lambda_0}{\lambda_0 - 1}, \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t) z_1'(t)}{z_1(t)} = \frac{1}{\lambda_0 - 1},$$

one has representation

$$f(t, z_0(t), z_1(t)) = \alpha_0 p(t) \varphi_1(z_1(t))[1 + o(1)] \quad as \quad t \uparrow \omega. \quad (5)$$

Moreover, under condition $(FN)_{\lambda_0}$ sign of second derivative of any $P_\omega(Y_0, Y_1, \lambda_0)$-solution of Eq. (1) in a left neighborhood of ω coincides with the value α_0. Then taking into account (2) , we have

$$\alpha_0 \mu_1 > 0 \quad \text{for} \quad Y_1 = \pm\infty \quad \text{and} \quad \alpha_0 \mu_1 < 0 \quad \text{for} \quad Y_1 = 0. \quad (6)$$

Main Results

1. Auxiliary statements

We choose a number $b \in \Delta_{Y_0}$ such that the inequality

$$|b| < 1 \quad \text{for} \quad Y_1 = 0, \quad b > 1 \ (b < -1) \quad \text{for} \quad Y_1 = +\infty \ (Y_1 = -\infty)$$

is respected and put

$$\Delta_{Y_1}(b) = [b, Y_1[\quad \text{if} \quad \Delta_{Y_1} \quad \text{is a left neighborhood of} \quad Y_1,$$

$$\Delta_{Y_1}(b) =]Y_1, b] \quad \text{if} \quad \Delta_{Y_1} \quad \text{is a right neighborhood of} \quad Y_1.$$

$$\Phi_1 : \Delta_{Y_1}(b) \longrightarrow \mathbb{R}, \quad \Phi_1(w) = \int_B^w \frac{ds}{\varphi_1(s)}, \quad B = \begin{cases} b & \text{if} \int_b^{Y_1} \frac{ds}{\varphi_1(s)} = \pm\infty, \\ Y_1 & \text{if} \int_b^{Y_1} \frac{ds}{\varphi_1(s)} = const, \end{cases}$$

$$I_1(t) = \int_A^t p(\tau)\,d\tau, \qquad A = \begin{cases} a & \text{if} \quad \int_a^\omega p(\tau)\,d\tau = \pm\infty, \\ \omega & \text{if} \quad \int_a^\omega p(\tau)\,d\tau = const, \end{cases}$$

$$Y^{[1]}(t) = \Phi_1^{-1}\left(\alpha_0 I_1(t)\right), \quad \mu_3 = sign\,\varphi_1(w) \quad \text{for} \quad w \in \Delta_{Y_1}.$$

Note that the function Φ_1 retains its sign on Δ_{Y_1}, tends either to 0 or to $\pm\infty$ as $w \to Y_1$, and increases on Δ_{Y_1} due to $\Phi_1'(w) > 0$. Therefore, it has an inverse function $\Phi_1^{-1} : \Delta_{Z_1} \to \Delta_{Y_1}$, where, due to the second of conditions (4) and the increase Φ_1^{-1}

$$Z_1 = \lim_{\substack{w \to Y_1 \\ w \in \Delta_{Y_1}}} \Phi_1(w) \in \{0, \pm\infty\}, \tag{7}$$

$$\Delta_{Z_1} = \begin{cases} [z_1, Z_1[& \text{if} \quad \Delta_{Y_1} \quad \text{is a left neighborhood of} \quad Y_1, \\ ]Z_1, z_1] & \text{if} \quad \Delta_{Y_1} \quad \text{is a right neighborhood of} \quad Y_1, \end{cases} \qquad z_1 = \Phi_1(b).$$

Definition 3. *Let $f : \Delta_{Y_1} \longrightarrow \mathbb{R}\backslash\{0\}$ be a twice continuously differentiable function. We will say that $f \in \Gamma(Y_1, Z_1)$ if it satisfies the following conditions*

$$f'(w) \neq 0, \quad \lim_{\substack{w \to Y_1 \\ w \in \Delta_{Y_1}}} f(w) = Z_1, \quad Z_1 = \left[\begin{array}{ll} or & 0, \\ eigther & \pm\infty, \end{array}\right.$$

$$\lim_{\substack{w \to Y_1 \\ w \in \Delta_{Y_1}}} \frac{f''(w) f(w)}{(f'(w))^2} = 1.$$

First of all, we note that, by virtue of definition 3, any function from $\Gamma(Y_1, Z_1)$-class is rapidly varying as $w \to Y_1$.

In [2] using the properties of functions from the class Γ introduced and studied in detail in the monograph Bingham N.H., Goldie C.M., Teugels J.L. [3] (Chapter 3, item 3.10), the following auxiliary assertions about the properties of functions from the class $\Gamma(Y_1, Z_1)$ were established.

Lemma 1. *If $f \in \Gamma(Y_1, Z_1)$ then there exists a continuous function $g : \Delta_{Y_1} \longrightarrow \mathbb{R} \setminus \{0\}$, called complementary to f, such that*

$$\lim_{\substack{w \to Y_1 \\ w \in \Delta_{Y_1}}} \frac{f(w + ug(w))}{f(w)} = e^u \quad \textit{for any} \quad u \in \mathbb{R},$$

moreover, the complementary function is uniquely determined up to functions equivalent as $w \to Y_1$*, for which, for example, one of the following functions*

$$\frac{\int\limits_W^w \left(\int\limits_W^t f(u)\,du \right) dt}{\int\limits_W^w f(x)\,dx} \sim \frac{\int\limits_W^w f(x)\,dx}{f(w)} \sim \frac{f(w)}{f'(w)} \sim \frac{f'(w)}{f''(w)} \quad as \quad w \to Y_1,$$

where

$$W = \begin{cases} z_1 \quad if \quad \lim\limits_{\substack{w \to Y_1 \\ w \in \Delta_{Y_1}}} f(w) = \pm\infty, \\ Y_1 \quad if \quad \lim\limits_{\substack{w \to Y_1 \\ w \in \Delta_{Y_1}}} f(w) = 0, \end{cases}$$

can be chosen.

Lemma 2.

1. *If* $f \in \Gamma(Y_1, Z_1)$ *with complementary function* g *then* $\lim\limits_{\substack{w \to Y_1 \\ w \in \Delta_{Y_1}}} \frac{g(w)}{w} = 0$.

2. *If* $f \in \Gamma(Y_1, Z_1)$ *with complementary function gthen for any continuous function* $u : \Delta_{Y_1} \longrightarrow \mathbb{R}$ *that satisfies the conditions*

$$\lim_{\substack{w \to Y_1 \\ w \in \Delta_{Y_1}}} u(w) = u_0 \in \mathbb{R}, \qquad \lim_{\substack{w \to Y_1 \\ w \in \Delta_{Y_1}}} f(w + u(w)g(w)] = Z_1,$$

there is a limit relation

$$\lim_{\substack{w \to Y_1 \\ w \in \Delta_{Y_1}}} \frac{f\left(w + u(w)g(w)\right)}{f(w)} = e^{u_0}.$$

Lemma 3. *If* $f \in \Gamma(Y_1, Z_1)$ *strictly monotone with complementary function gthen its inverse function* $f^{-1} : \Delta_{Z_1} \longrightarrow \Delta_{Y_1}$ *is slowly varying at* $z \to Z_1$ *and satisfies the limit relation*

$$\lim_{\substack{z \to Z_1 \\ z \in \Delta_{Z_1}}} \frac{f^{-1}(\lambda z) - f^{-1}(z)}{g(f^{-1}(z))} = \ln \lambda \quad for\ any \quad \lambda > 0,$$

moreover for any given $\Lambda > 1$ *limit relation is satisfied uniformly in* $\lambda \in \left[\frac{1}{\Lambda}, \Lambda\right]$.

Note also, it follows from the Representation Theorem for Γ ([3],Chapter 3, item 3.10, position d) that for a function $f \in \Gamma(Y_1, Z_1)$ there exists a continuously differentiable function $f_1 \in \Gamma(Y_1, Z_1)$ such that

$$\lim_{\substack{w \longrightarrow Y_1 \\ w \in \Delta_{Y_1}}} \frac{f(w)}{f_1(w)} = 1 \quad \text{and} \quad \lim_{\substack{w \longrightarrow Y_1 \\ w \in \Delta_{Y_1}}} \frac{w f_1(w)}{f_1(w)} = \pm\infty.$$

2. Main results

Theorem 1. *Let $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$ and let the function f satisfies condition $(FN1)_{\lambda_0}$. Then, for the existence of $P_\omega(Y_0, Y_1, \lambda_0)$- solutions of the differential equation (1), it is necessary that the conditions (2), (6),*

$$\mu_0\mu_1\lambda_0(\lambda_0 - 1)\pi_\omega(t) > 0, \quad \alpha_0\mu_3 I_1(t) < 0 \quad \textit{for} \quad t \in [a, \omega[\tag{8}$$

$$\alpha_0 \lim_{t\uparrow\omega} I_1(t) = Z_1, \tag{9}$$

$$\lim_{t\uparrow\omega} \frac{I_1'(t)\pi_\omega(t)}{I_1(t)} = \pm\infty, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)p(t)\varphi_1(Y^{[1]}(t))}{Y^{[1]}(t)} = \frac{\alpha_0}{\lambda_0 - 1} \tag{10}$$

are hold.

Moreover, each solution of this kind admits the asymptotic representations as $t \uparrow \omega$

$$y(t) = \frac{(\lambda_0 - 1)}{(\lambda_0)} Y^{[1]}(t)\pi_\omega(t)(1 + o(1)), \quad y'(t) = Y^{[1]}(t)[1 + o(1)]. \tag{11}$$

Proof of Theorem 1. Let $\lambda_0 \in \mathbf{R} \setminus \{0, 1\}$ and $y : [t_0, \omega[\to \Delta_{Y_0}$ be an arbitrary $P_\omega(Y_0, Y_1, \lambda_0)-$ solution of Eq. (1.1). Then there is a number $t_1 \in [t_0, \omega[$ such that $y^{(k)}(t) \neq 0$ $(k = 0, 1, 2)$, $\operatorname{sign} y^{(k)}(t) = \mu_k$ $(k = 0, 1)$ for $t \in [t_1, \omega[$. Moreover, from the equality

$$\left(\frac{y(t)}{y'(t)}\right)' = 1 - \frac{y(t)y''(t)}{(y'(t))^2}$$

and conditions (3), the definition of the $P_\omega(Y_0, Y_1, \lambda_0)-$ solution immediately implies that

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)y'(t)}{y(t)} = \frac{\lambda_0}{\lambda_0 - 1}, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)y''(t)}{y'(t)} = \frac{1}{\lambda_0 - 1}. \tag{12}$$

From this, in particular, it follows that the first of the sign representations (8) holds. Due to (12) and the condition $(FN1)_{\lambda_0}$ which the function f satisfies from (1) we have

$$y''(t) = \alpha_0 p(t)\varphi_1(y'(t))[1 + o(1)] \quad \text{as} \quad t \uparrow \omega \tag{13}$$

or

$$\frac{y''(t)}{\varphi_1(y'(t))} = \alpha_0 p(t)[1 + o(1)] \quad \text{as} \quad t \uparrow \omega. \tag{14}$$

Integration of (14) on the interval from t_1 to t leads to the limiting equality

$$\int_{t_1}^{t} \frac{y''(\tau)\, d\tau}{\varphi_1(y'(\tau))} = \alpha_0 \int_{t_1}^{t} p(\tau)\, d\tau\, [1 + o(1)] \quad \text{as} \quad t \uparrow \omega$$

or by virtue of the definition of the limits of integration A and B

$$\Phi_1(y'(t)) = \alpha_0 I_1(t)[1 + o(1)] \quad \text{as} \quad t \uparrow \omega. \tag{15}$$

It follows from condition (4) that the function φ_1 together with its derivative of the first order are rapidly varying as $y' \to Y_1$, because

$$\lim_{\substack{y' \longrightarrow Y_1 \\ y' \in \Delta_{Y_1}}} \frac{y' \varphi_1'(y')}{\varphi_1(y')} = \pm\infty, \quad \lim_{\substack{y' \longrightarrow Y_1 \\ y' \in \Delta_{Y_1}}} \frac{y' \varphi_1''(y')}{\varphi_1'(y')} = \pm\infty. \tag{16}$$

Also from (4) as $y' \to Y_1$ the equivalence $\frac{\varphi_1'(y')}{\varphi_1(y')} \sim \frac{\varphi_1''(y')}{\varphi_1'(y')}$ follows. In addition, taking to account the L'Hopital rule in the form of Stolz, we can assert that

$$\lim_{\substack{y' \longrightarrow Y_1 \\ y' \in \Delta_{Y_1}}} \frac{\Phi_1(y')}{\frac{1}{\varphi_1'(y')}} = \lim_{\substack{y' \longrightarrow Y_1 \\ y' \in \Delta_{Y_1}}} \frac{\frac{1}{\varphi_1(y')}}{-\frac{\varphi_1''(y')}{(\varphi_1'(y'))^2}} = -\lim_{\substack{y' \longrightarrow Y_1 \\ y' \in \Delta_{Y_1}}} \frac{(\varphi_1(y'))^2}{\varphi_1''(y')\varphi_1(y')}, \tag{17}$$

hence

$$\begin{gathered} \Phi_1(y') \sim -\frac{1}{\varphi_1'(y')} \quad \text{as} \quad y' \to Y_1, \\ \Phi_1(y')\varphi_1'(y') < 0 \quad \text{for} \quad y' \in \Delta_{Y_1}. \end{gathered} \tag{18}$$

A consequence of conditions (18), (15) is the second of the inequalities (8). Condition (17) implies as $y' \to Y_1$ fulfillment of the equivalences

$$\frac{\Phi_1'(y')}{\Phi_1(y')} = \frac{\frac{1}{\varphi_1(y')}}{\Phi_1(y')} \sim -\frac{\varphi_1'(y')}{\varphi_1(y')}, \quad \frac{\Phi_1''(y')\Phi_1(y')}{(\Phi_1'(y'))^2} = \frac{-\frac{\varphi_1'(y')}{\varphi_1^2(y')}\Phi_1(y')}{\frac{1}{\varphi_1^2(y')}} \sim 1. \tag{19}$$

Hence taking into account the lemma 2.14 (see [2, Chap. II, Sec. 2.3, P. 54]) it follows that the function Φ_1 belongs to the class $\Gamma(Y_1, Z_1)$ with a complementary function g, for which one can choose one of the equivalent functions

$$\frac{\Phi_1'(y')}{\Phi_1''(y')} \sim \frac{\Phi_1(y')}{\Phi_1'(y')} \sim -\frac{\varphi_1(y')}{\varphi_1'(y')} \quad \text{as} \quad y' \to Y_1.$$

Further from (14), (15), (18) by virtue of (9) and (16) the first of the conditions (10) follows.

Because the function Φ_1 belongs to the class $\Gamma(Y_1, Z_1)$ and complementary to it can be chosen as $g(y') = -\frac{\varphi_1(y')}{\varphi_1'(y')}$. From the definition of Z_1, μ_3, the second of sign conditions (8) $\Phi_1^{-1}(\alpha_0 I_1(t)) \in \Delta_{Y_1}$ as $t \in [t_0, \omega[$ and (9) follow. Therefore, based on the lemma 3 we have the limit equality

$$\lim_{t\uparrow\omega} \frac{\Phi_1^{-1}(\alpha_0 I_1(t)[1+o(1)]) - \Phi_1^{-1}(\alpha_0 I_1(t))}{-\frac{\varphi_1\left(\Phi_1^{-1}(\alpha_0 I_1(t))\right)}{\varphi_1'\left(\Phi_1^{-1}(\alpha_0 I_1(t))\right)}} =$$

$$= \lim_{\substack{z\to Z_1\\ z\in\Delta_{Z_1}}} \frac{\Phi_1^{-1}(z[1+o(1)]) - \Phi_1^{-1}(z)}{-\frac{\varphi_1(z)}{\varphi_1'(z)}} = 0,$$

which we can rewrite in the form

$$\Phi_1^{-1}(\alpha_0 I_1(t)[1+o(1)]) = \Phi_1^{-1}(\alpha_0 I_1(t)) + \frac{\varphi_1\left(\Phi_1^{-1}(\alpha_0 I_1(t))\right)}{\varphi_1'\left(\Phi_1^{-1}(\alpha_0 I_1(t))\right)} o(1) \quad \text{as} \quad t \uparrow \omega.$$

Thus, the second of (11) is established, since

$$\lim_{t\uparrow\omega} \frac{Y^{[1]}(t)\varphi_1'\left(Y^{[1]}(t)\right)}{\varphi_1\left(Y^{[1]}(t)\right)} = \lim_{\substack{y'\to Y_1\\ y'\in\Delta_{Y_1}}} \frac{y'\varphi_1'(y')}{\varphi_1(y')} = \pm\infty.$$

Invoking the first of (11) from (12), we obtain the first of the relations (11). Now we write (13) in the form

$$y''(t) = \alpha_0 p(t)\varphi_1\left(Y^{[1]}(t) + \frac{\varphi_1\left(Y^{[1]}(t)\right)}{\varphi_1'\left(Y^{[1]}(t)\right)}\right)[1+o(1)] \quad \text{as} \quad t\uparrow\omega. \tag{20}$$

Then, as a complementary to the function $\varphi_1 \in \Gamma(Y_1, Z_1)$, we choose $g(y') = \frac{\varphi_1(y')}{\varphi_1'(y')}$. Then, taking into account that $\lim\limits_{t\uparrow\omega} Y^{[1]}(t) = Y_1$, $Y^{[1]}(t) \in \Delta_{Y_1}$ at

$t \in [t_0, \omega]$, we obtain

$$\lim_{t\uparrow\omega} \frac{\varphi_1\left(Y^{[1]}(t) + \dfrac{\varphi_1(Y^{[1]}(t))}{\varphi_1'(Y^{[1]}(t))}o(1)\right)}{\varphi_1\left(Y^{[1]}(t)\right)} = \lim_{\substack{y\to Y_1 \\ y'\in\Delta_{Y_1}}} \frac{\varphi_1\left(y' + \dfrac{\varphi_1(y')}{\varphi_1'(y')}o(1)\right)}{\varphi_1(y')} = 1,$$

which in turn leads to

$$\varphi_1\left(Y^{[1]}(t) + \frac{\varphi_1(Y^{[1]}(t))}{\varphi_1'(Y^{[1]}(t))}o(1)\right) = \varphi_1\left(Y^{[1]}(t)\right)[1+o(1)] \quad \text{as} \quad t\uparrow\omega.$$

Therefore, relation (20) takes the form

$$y''(t) = \alpha_0 p(t)\varphi_1\left(Y^{[1]}(t)\right)[1+o(1)] \quad \text{as} \quad t\uparrow\omega.$$

From the last representation, taking into account the second of the limit equalities (12), we obtain he second of conditions (10).
The theorem is proved.

Conclusion

In this paper, we consider the differential equation of the second order, which is asymptotically close as $t\uparrow\omega$ to a two-term equation with rapidly varying with respect to the derivative component. Received necessary conditions for the existence of $P_\omega(Y_0, Y_1, \lambda_0)-$ solutions, as well as asymptotic representations for such solutions and their derivatives. To obtain sufficient conditions for existence solutions of this class, it is required to involve the results of the work [4]. It is also possible to refine the asymptotics of $P_\omega(Y_0, Y_1, \lambda_0)-$solutions in terms of functions φ_1, $Y^{[1]}$, π_ω.

Кусік Л. І.

Необхідні умови існування $P_\omega(Y_0, Y_1, \lambda_0)$- розв'язків диференціального рівняння другого порядку з щвидко змінною нелінійністю

Резюме

Розглядаємо диференціальне рівняння другого порядку загального виду $y'' = f(t, y, y')$, де $f : [a, \omega[\times\Delta_{Y_0}\times\Delta_{Y_1} \longrightarrow \mathbf{R}$ — неперервна функція, $-\infty < a < \omega \le +\infty$, Δ_{Y_i} — односторонній окіл Y_i, $Y_i \in \{0, \pm\infty\}$ $(i \in \{0,1\})$. При певних умовах на функцію f це рівняння може бути подане близьким до двочленного, а саме $y'' = \alpha_0 p(t)\varphi_1(y')(1+o(1))$ при $t\uparrow\omega$, де φ_1 — швидко змінна при $y'\to Y_1$ функція. Знайдено необхідні умови

існування розв'язків, для яких $\lim\limits_{t\uparrow\omega} y^{(i)}(t) = Y_i$ $(i \in \{0,1\})$, $\lim_{t\uparrow\omega} \dfrac{[y'(t)]^2}{y(t)y''(t)} = \lambda_0$, т. з. $P_\omega(Y_0, Y_1, \lambda_0)$-розв'язків. Такого типу розв'язки раніше було введено в роботах Євтухова В. М., Белозерової М. О. при вивченні двочленного рівняння $y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y')$, де $\alpha_0 \in \{-1, 1\}$,$p : [a, \omega[\longrightarrow]0, +\infty[$–неперервна функція, $\varphi_i : \Delta_{Y_i} \longrightarrow]0, +\infty[$ $(i = 0, 1)$ — неперервні правильно змінні при $z \to Y_i$ $(i = 0, 1)$ функції порядків σ_i $(i = 0, 1)$, причому $\sigma_0 + \sigma_1 \neq 1$. Далі, у дослідженнях Євтухова В. М., Чернікової А. Г. для рівняння $y'' = \alpha_0 p(t)\varphi_0(y)$ встановлено необхідні, достатні умови існування, а також асимптотичні при $t \uparrow \omega$ зображення $P_\omega(Y_0, Y_1, \lambda_0)$-розв'язків у випадку, коли φ_0 — швидко змінна при $y \to Y_0$ функція.
Ключові слова: двочленне диференціальне рівняння, $P_\omega(Y_0, Y_1, \lambda_0)$-розв'язок, асимптотичні зображення розв'язків, швидко змінна функція, одно-, двопараметрична сім'я розв'язків.

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