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THE DYNAMICAL PROBLEM ON ACTING DISTRIBUTED LOAD ON THE ELASTIC LAYER

The wave field of an elastic half-layer is constructed, when a dynamic normal load distributed over a rectangular area acts on upper face at the initial moment of time. The lower face of the half-layer is rigidly fixed to the foundation, and the side border is in the conditions of a smooth contact. The method of decomposing the system of motion equations into a system of equations and an independently solvable equation is used, this approach was proposed by Popov G. Ya. Laplace and Fourier integral transformations are applied directly to the motion equations and boundary conditions, which reduces the problem to a vector one-dimensional boundary value problem, which is solved by the matrix differential calculus method. The output displacements are obtained using inverse integral transformations. The case of steady oscillations was considered and the amplitude of vertical displacement occurring in the layer was analyzed depending on the shape of the distributed load section, the material of the layer medium and the values of the natural frequency of the layer oscillations.
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Introduction

Dynamic problems of the elasticity theory are solved for during construction to obtain the displacements in elastic bodies. Displacements lead to damage or deformation of the structure. Therefore, in mathematical physics, many authors solve the problems of the elasticity theory. Popov G. Ya. developed the method of presenting the Lame equations through two jointly and one separately solved equations in his work [7]. The exact solution for the mixed problem of the elasticity theory was found in [8]. Also, Popov G. Ya., in collaboration with Vaysfeld N. D [10]., found a solution to the Lamb problem using this method. In [15], a solution was found for semi-homogeneous and non-homogeneous problems of the elasticity theory for a semi-infinite layer in a static formulation. Dynamical problem for an elastic quarter space was found by Fesenko A. A., Bondarenko K. S. in [3]. Dynamical stresses in elastic half-space were analysed in [16] by Winfried Schepers. Plane contact problem on

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the pressure of a stamp with a rectangular base on a rough elastic halfspace was considered in [12]. Also, solution methods of dynamic problems have been described at book [11]. Some problems of the elasticity theory for an elastic layer were solved in [1; 5; 6]. Also, a solution was found for the dynamical problem for the infinite elastic layer with a cylindrical cavity by Fesenko A. A. in [2].

The aim of this work is to obtain the exact formulas for displacements that appear in a elastic layer when a dynamic compressive load acts on upper faces.

Main Results

1. Statement of the problem. Consider the elastic layer $x > 0$, $-\infty < y < \infty$, $0 < z < h$. The dynamic normal load is acting on the boundary of the layer $z = h$ along the rectangular zone $0 \le x \le A$, $-B \le y \le B$. The smooth contact conditions are set at the side boundary $x = 0$. The boundary $z = 0$ is rigidly fixed. It is necessary to find displacements of the points of the layer $U(x, y, z, t)$, $V(x, y, z, t)$, $W(x, y, z, t)$ with zero initial conditions. The statement, leads to the following boundary conditions

$$\begin{aligned}
&\sigma_z(x, y, h, t) = -p(x, y)P(t),\ 0 \le x \le A;\ -B \le y \le B,\\
&\tau_{zx}(x, y, h, t) = 0,\ \tau_{zy}(x, y, h, t) = 0,\\
&U(x, y, 0, t) = V(x, y, 0, t) = W(x, y, 0, t) = 0,\\
&U(0, y, z, t) = \frac{\partial V(0, y, z, t)}{\partial x} = \frac{\partial W(0, y, z, t)}{\partial x} = 0.
\end{aligned} \tag{1}$$

The motion equations in vector form have the form [7]

$$\Delta(U, V, W) + \frac{2}{\kappa - 1}\left(\frac{\partial\Theta}{\partial x}, \frac{\partial\Theta}{\partial y}, \frac{\partial\Theta}{\partial z}\right) = \frac{\rho}{G}\left(\frac{\partial^2 U}{\partial t^2}, \frac{\partial^2 V}{\partial t^2}, \frac{\partial^2 W}{\partial t^2}\right) \tag{2}$$

Where Δ — Laplace operator, $\kappa = 3 - 4\mu$, μ — Poisson's ratio, $\Theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ — volume expansion, ρ — material density, G — shear modulus.

To obtain a solution to the given problem, it is necessary to obtain a solution for the dynamic force concentrated at an arbitrary point on the boundary $z = h$

$$p(x, y) = -\delta(x - a)\delta(y - b),$$

where δ – Dirac function, and then distribute it over the required area.

Let's introduce new functions [7]

$$Z(x,y,z)=\frac{\partial}{\partial x}U(x,y,z)+\frac{\partial}{\partial y}V(x,y,z),$$
$$\widetilde{Z}(x,y,z)=\frac{\partial}{\partial x}V(x,y,z)-\frac{\partial}{\partial y}U(x,y,z).$$

Then the system of motion equations (2) and boundary conditions (1) taking into account the new functions will be rewritten in the form:

$$\begin{cases}\Delta W+\dfrac{2}{\kappa-1}\dfrac{\partial}{\partial z}\left(Z+\dfrac{\partial W}{\partial z}\right)=\dfrac{(\kappa-1)}{(\kappa+1)}\dfrac{\rho}{G}\dfrac{\partial^2 W}{\partial t^2},\\ \Delta Z+\dfrac{2}{\kappa-1}\nabla_{xy}\left(Z+\dfrac{\partial W}{\partial z}\right)=\dfrac{\rho}{G}\dfrac{\partial^2 Z}{\partial t^2},\end{cases}\tag{3}$$

$$\Delta\widetilde{Z}=\frac{\partial^2\widetilde{Z}}{\partial t^2},\tag{4}$$

$$\begin{gathered}\nabla_{xy}W(x,y,h,t)+\frac{\partial}{\partial z}Z(x,y,h,t)=0,\\ (3-\kappa)Z(x,y,h,t)+(1+\kappa)\frac{\partial}{\partial z}W(x,y,h,t)=-\frac{\kappa-1}{G}\delta(x-a)\delta(y-b)P(t),\\ Z(x,y,0,t)=\widetilde{Z}(x,y,0,t)=W(x,y,0,t)=0,\\ \frac{\partial}{\partial z}\tilde{Z}(x,y,h,t)=0,\\ \frac{\partial}{\partial x}Z(0,y,z,t)=\frac{\partial}{\partial x}W(0,y,z,t)=\tilde{Z}(0,y,z,t)=0,\end{gathered}\tag{5}$$

where $\nabla_{xy}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

The initial boundary value problem takes the form (3)–(5) under the initial conditions

$$\left[W,Z,\widetilde{Z}\right]\Big|_{t=0}=0\ \ \frac{\partial}{\partial t}\left[W,Z,\widetilde{Z}\right]\Big|_{t=0}=0.$$

After finding the functions $W,Z,\widetilde{Z}$ to find the displacements U and V the Poisson equation should be solved

$$\nabla_{xy}U=\frac{\partial}{\partial x}Z-\frac{\partial}{\partial y}\widetilde{Z},\ \nabla_{xy}V=\frac{\partial}{\partial y}Z+\frac{\partial}{\partial x}\widetilde{Z}.\tag{6}$$

2. Reduction the problem to a vector one-dimensional problem. The *cos* - Fourier transform with respect to the variable x, the Fourier transform with respect to the variable y and the Laplace transform of the variable t

with parameters α, β and p, respectively are successively applied to the (3)–(4).

$$\begin{bmatrix} W_{\alpha\beta p}(z) \\ Z_{\alpha\beta p}(z) \end{bmatrix} = \int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}, \begin{bmatrix} W(x,y,z,t) \\ Z(x,y,z,t) \end{bmatrix} e^{i\beta y}\cos\alpha x\, e^{-pt}\, dy\, dx\, dt$$

where $N^2 = \alpha^2 + \beta^2$.

The function $\widetilde{Z}_{\alpha\beta p}(z)$ satisfies the homogeneous problem

$$\widetilde{Z}''_{\alpha\beta p}(z) - (N^2 + p^2)\widetilde{Z}_{\alpha\beta p}(z) = 0,\ 0 < z < h,\ \widetilde{Z}'_{\alpha\beta p}(h) = 0,\ \widetilde{Z}_{\alpha\beta p}(0) = 0 \quad (7)$$

and therefore $\widetilde{Z}(x,y,z,t) \equiv 0$.

3. A case of steady-state oscillations. To consider a steady-state oscillations suppose that load applied across the area $0 < x < A;\ -B < y < B$ over the plane $X0Y$ changes according to the harmonic law $P(t) = e^{i\omega t}$ and $p(x,y) = P$, where P – constant intensity of the load, ω – is a natural frequency of vibrations. In this case, substituting into the system of equations and boundary conditions $p = i\omega$ according to the [4].

Let's introduce the values

$$k_1^2 = \frac{\omega^2\rho}{G}, k_2^2 = \frac{(\kappa-1)}{\kappa+1}\frac{\omega^2\rho}{G}, \quad (8)$$

where $k_1,\ k_2$ – the wave numbers.

The system of equations (3) and boundary conditions (5) take the form

$$\begin{cases} W''_{\alpha\beta}(z;k_1,k_2) + \dfrac{2}{\kappa+1}Z'_{\alpha\beta}(z;k_1,k_2) - N^2\dfrac{\kappa-1}{\kappa+1}W_{\alpha\beta}(z;k_1,k_2) + \\ \qquad + k_2^2 W_{\alpha\beta}(z;k_1,k_2) = 0, \\ Z''_{\alpha\beta}(z;k_1,k_2) - \dfrac{2}{\kappa-1}N^2 W'_{\alpha\beta}(z;k_1,k_2) - N^2\dfrac{\kappa+1}{\kappa-1}Z_{\alpha\beta}(z;k_1,k_2) + \\ \qquad + k_1^2 Z_{\alpha\beta}(z;k_1,k_2) = 0, \end{cases} \quad (9)$$

$$\begin{gathered} -N^2 W_{\alpha\beta}(h;k_1,k_2) + Z'_{\alpha\beta}(h;k_1,k_2) = 0, \\ (3-\kappa)Z_{\alpha\beta}(h;k_1,k_2) + (\kappa+1)W'_{\alpha\beta}(h;k_1,k_2) = -\frac{\kappa-1}{G}\cdot\cos\alpha a e^{ib\beta}\cdot P, \\ Z_{\alpha\beta}(0;k_1,k_2) = W_{\alpha\beta}(0;k_1,k_2) = 0, \\ N^2 = \alpha^2 + \beta^2. \end{gathered} \quad (10)$$

To reduce problems (9) (10) to a vector one-dimensional one, an unknown transform vector of displacements is introduced

$$\vec{\mathbf{y}}(z;k_1,k_2)=\begin{pmatrix}W_{\alpha\beta}(z;k_1,k_2)\\ Z_{\alpha\beta}(z;k_1,k_2)\end{pmatrix}$$

as well as matrices

$$\mathbf{I}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ \mathbf{Q}=\begin{pmatrix}0 & \frac{2}{\kappa+1}\\ \frac{-2N^2}{\kappa-1} & 0\end{pmatrix},\ \mathbf{P}=\begin{pmatrix}\frac{\kappa-1}{\kappa+1} & 0\\ 0 & \frac{\kappa+1}{\kappa-1}\end{pmatrix},\ \mathbf{T}=\begin{pmatrix}k_2^2 & 0\\ 0 & k_1^2\end{pmatrix}.$$

So, the system (9) and boundary conditions (10) takes the form

$$\begin{cases}\mathrm{L}_2\vec{\mathbf{y}}(z;k_1,k_2)=0,\ 0<z<h,\\ \quad \mathbf{U}_0[\vec{\mathbf{y}}(0;k_1,k_2)]=\mathbf{\Theta}_0,\\ \quad \mathbf{U}_1[\vec{\mathbf{y}}(h;k_1,k_2)]=\mathbf{\Theta}_1,\end{cases}\tag{11}$$

where the differential operator L_2 has the form

$$\mathrm{L}_2\vec{\mathbf{y}}(z;k_1,k_2)=\mathbf{I}\vec{\mathbf{y}}''(z;k_1,k_2)+\mathbf{Q}\vec{\mathbf{y}}'(z;k_1,k_2)-N^2\mathbf{P}\vec{\mathbf{y}}(z;k_1,k_2)+\mathbf{T}\vec{\mathbf{y}}(z;k_1,k_2).$$

Let's enter matrices and vectors

$$\mathbf{A}=\begin{pmatrix}-N^2 & 0\\ 0 & (3-\kappa)\end{pmatrix},\ \mathbf{B}=\begin{pmatrix}0 & 1\\ (1+\kappa) & 0\end{pmatrix},$$

$$\mathbf{\Theta}_0=(0,0)^T,\ \mathbf{\Theta}_1=(0,-\frac{P(\kappa-1)}{G}\cos\alpha a e^{ib\beta})^T,$$

where symbol T means transported vector. Edge functionals are

$$\mathbf{U}_0[\vec{\mathbf{y}}]=\mathbf{I}\vec{\mathbf{y}}(0;k_1,k_2),$$

$$\mathbf{U}_1[\vec{\mathbf{y}}]=\mathbf{A}\vec{\mathbf{y}}(h;k_1,k_2)+\mathbf{B}\vec{\mathbf{y}'}(h;k_1,k_2).$$

The solution of the vector equation (11) is built on the basis of the solution of the matrix equation $\mathrm{L}_2\left[\mathbf{Y}(z)\right]=0$. Substitution $\mathbf{Y}(z)=e^{Nz}\mathbf{I}$ is made to form the characteristic matrix $\mathbf{M}(s)=\mathbf{I}s^2+\mathbf{Q}s-N^2\mathbf{P}+\mathbf{T}$. The inverse matrix has the form

$$\mathbf{M^{-1}(s)}=\frac{1}{\prod_{i=1}^4(s-s_i)}\begin{pmatrix}s^2-N^2\frac{\kappa+1}{\kappa-1}+k_2^1 & -\frac{2s}{\kappa+1}\\ \frac{2s}{\kappa-1}N^2 & s^2-N^2\frac{\kappa-1}{\kappa+1}+k_2^2\end{pmatrix},$$

$$s_1 = \sqrt{N^2 - k_2^2},\ s_2 = -\sqrt{N^2 - k_2^2},\ s_3 = \sqrt{N^2 - k_1^2},\ s_4 = -\sqrt{N^2 - k_1^2}.$$

Here s_i $(i = \overline{1,4})$ are the roots of the characteristic equation $\det[\mathbf{M}(s)] = 0$. The solution of the matrix equation is constructed according to the formula [9]

$$\mathbf{Y}(z) = \frac{1}{2\pi i} \oint\limits_C e^{sz} \mathbf{M}^{-1}(s) ds,$$

where C is a closed loop covering all zeros of the determinant of the matrix $\mathbf{M}(s)$. The residues at the poles s_1 and s_3 give an increasing solution that has the form

$$\mathbf{Y}_+(z; k_1, k_2) = -\frac{1}{2k_1^2} e^{\Delta_1 z} \begin{pmatrix} -\frac{(\kappa+1)N^2}{(\kappa-1)\Delta_1} & -1 \\ \frac{(\kappa+1)N^2}{(\kappa-1)} & \Delta_1 \end{pmatrix} - \frac{1}{2k_1^2} e^{\Delta_2 z} \begin{pmatrix} \frac{(\kappa+1)}{(\kappa-1)}\Delta_2 & 1 \\ -\frac{(\kappa+1)}{(\kappa-1)}N^2 & -\frac{N^2}{\Delta_2} \end{pmatrix}.$$

The residuals at the poles s_2 and s_4 give a solution that descends.

$$\mathbf{Y}_-(z; k_1, k_2) = -\frac{1}{2k_1^2} e^{-\Delta_1 z} \begin{pmatrix} \frac{(\kappa+1)N^2}{(\kappa-1)\Delta_1} & -1 \\ \frac{(\kappa+1)N^2}{(\kappa-1)} & -\Delta_1 \end{pmatrix} - \frac{1}{2k_1^2} e^{-\Delta_2 z} \begin{pmatrix} -\frac{(\kappa+1)}{(\kappa-1)}\Delta_2 & 1 \\ -\frac{(\kappa+1)}{(\kappa-1)}N^2 & \frac{N^2}{\Delta_2} \end{pmatrix},$$

where $\Delta_1 = \sqrt{N^2 - k_1^2},\ \Delta_2 = \sqrt{N^2 - k_2^2}$.

The solution of the vector equation (11) is constructed in the form

$$\vec{\mathbf{y}}(z) = \mathbf{\Psi}_0 \mathbf{\Theta}_0 + \mathbf{\Psi}_1 \mathbf{\Theta}_1,$$

where $\mathbf{\Psi}_i,\ i = 0,1$ - the fundamental basis matrices of the solutions, $\mathbf{\Theta}_i,\ i = 0,1$ - the right-hand parts of the boundary conditions.

The fundamental basis matrices is constructed through the fundamental system of solutions of the homogeneous differential equation (11), using the formulas $\mathbf{\Psi}_i = \mathbf{Y}_-(z)\mathbf{C}_i^0 + \mathbf{Y}_+(z)\mathbf{C}_i^1,\ i = 0,1.$ $\mathbf{C}_i^{0,1},$ — are matrices of unknown constants [9]. The matrices of unknown constants can be found from the relations by satisfying the boundary conditions $\mathbf{U}_i[\mathbf{\Psi}] = \delta_{ij}\mathbf{I}\ i,j = 0,1$

$$\mathbf{C}_1^1 = (\mathbf{U}_1[\mathbf{Y}_+(z)] - \mathbf{U}_1[\mathbf{Y}_-(z)] \cdot (\mathbf{U}_0[\mathbf{Y}_-(z)])^{-1} \cdot \mathbf{U}_0[\mathbf{Y}_-(z)])^{-1},$$

$$\mathbf{C}_1^0 = -(\mathbf{U}_0[\mathbf{Y}_-(z)])^{-1} \cdot \mathbf{U}_0[\mathbf{Y}_+(z)] \cdot \mathbf{C}_1^1,$$

$$\mathbf{U}_0[\mathbf{Y}_+(z)] = -\frac{1}{2k_1^2} \left(\begin{pmatrix} -\frac{(\kappa+1)N^2}{(\kappa-1)\Delta_1} & -1 \\ \frac{(\kappa+1)N^2}{(\kappa-1)} & \Delta_1 \end{pmatrix} + \begin{pmatrix} \frac{(\kappa+1)}{(\kappa-1)}\Delta_2 & 1 \\ -\frac{(\kappa+1)}{(\kappa-1)}N^2 & -\frac{N^2}{\Delta_2} \end{pmatrix} \right),$$

$$\mathbf{U}_0[\mathbf{Y}_-(z)] = -\frac{1}{2k_1^2}\left(\begin{pmatrix} \frac{(\kappa+1)N^2}{(\kappa-1)\Delta_1} & -1 \\ \frac{(\kappa+1)N^2}{(\kappa-1)} & -\Delta_1 \end{pmatrix} + \begin{pmatrix} -\frac{(\kappa+1)}{(\kappa-1)}\Delta_2 & 1 \\ -\frac{(\kappa+1)}{(\kappa-1)}N^2 & \frac{N^2}{\Delta_2} \end{pmatrix}\right),$$

$$\mathbf{U}_1[\mathbf{Y}_+(z)] = -\frac{1}{2k_1^2}\begin{pmatrix} \frac{\kappa+1}{\kappa-1}N^2\left(\left(\frac{N^2}{\Delta_1}+\Delta_1\right)e^{\Delta_1 h} - 2\Delta_2 e^{\Delta_2 h}\right) & \left(2N^2-k_1^2\right)e^{\Delta_1 h} - 2N^2 e^{\Delta_2 h} \\ (\kappa+1)\left(-2N^2 e^{\Delta_1 h} + \left(2N^2-k_1^2\right)e^{\Delta_2 h}\right) & (\kappa-1)\left(-2\Delta_1 e^{\Delta_1 h} + \frac{1}{\Delta_2}\left(2N^2-k_1^2\right)e^{\Delta_2 h}\right) \end{pmatrix},$$

$$\mathbf{U}_1[\mathbf{Y}_-(z)] = -\frac{1}{2k_1^2}\begin{pmatrix} \frac{\kappa+1}{\kappa-1}N^2\left(2\Delta_2 e^{-\Delta_2 h} - \left(\frac{N^2}{-\Delta_1}+\Delta_1\right)e^{-\Delta_1 h}\right) & \left(2N^2-k_1^2\right)e^{-\Delta_1 h} - 2N^2 e^{-\Delta_2 h} \\ (\kappa+1)\left(-2N^2 e^{-\Delta_1 h} + \left(2N^2-k_1^2\right)e^{-\Delta_2 h}\right) & (\kappa-1)\left(2\Delta_1 e^{-\Delta_1 h} - \frac{1}{\Delta_2}\left(2N^2-k_1^2\right)e^{-\Delta_2 h}\right) \end{pmatrix}.$$

Taking into account that $\mathbf{U}_0[\mathbf{Y}_-(z)]^{-1}\mathbf{U}_0[\mathbf{Y}_+(z)] = -\mathbf{I}$ we get that $\mathbf{C}_1^1 = \mathbf{C}_1^0$. Since $\mathbf{\Theta}_0 = (0,0)^T$ then $\mathbf{\Psi_0}$ is not of interest. Matrix $\mathbf{\Psi_1}$ has a form

$$\mathbf{\Psi_1} = -\frac{1}{2k_1^2}\begin{pmatrix} \frac{\kappa+1}{\kappa-1}\left(\Delta_2\sinh\Delta_2 z - \frac{N^2}{\Delta_1}\sinh\Delta_1 z\right) & \frac{\kappa+1}{\kappa-1}N^2\left(\cosh\Delta_1 z - \cosh\Delta_2 z\right) \\ \cosh\Delta_2 z - \cosh\Delta_1 z & \Delta_1\sinh\Delta_1 z - \frac{N^2}{\Delta_2}\sinh\Delta_2 z \end{pmatrix}\cdot\mathbf{C}_1^1.$$

After simplification, expressions for the transformants were found

$$\begin{aligned} W_{\alpha\beta}(z;k_1,k_2) = &-\frac{\cos\alpha a e^{ib\beta}}{G}\cdot P\frac{\Delta_2}{\widetilde{\Delta}}\big[\left(\Delta_1\Delta_2\sinh\Delta_2 z - N^2\sinh\Delta_1 z\right)\times \\ &\times\left(2N^2\cosh\Delta_2 h - (2N^2-k_1^2)\cosh\Delta_1 h\right) + \\ &+N^2\left(\cosh\Delta_2 z - \cosh\Delta_1 z\right)\times \\ &\times\left((2N^2-k_1^2)\sinh\Delta_1 h - 2\Delta_1\Delta_2\sinh\Delta_2 h\right)\big], \\ Z_{\alpha\beta}(z;k_1,k_2) = &-\frac{\cos\alpha a e^{ib\beta}}{G}\cdot P\frac{N^2}{\widetilde{\Delta}}[\Delta_1\Delta_2\left(\cosh\Delta_1 z - \cosh\Delta_2 z\right)\times \\ &\times\left(2N^2\cosh\Delta_2 h - (2N^2-k_1^2)\cosh\Delta_1 h\right) + \\ &+\left(\Delta_1\Delta_2\sinh\Delta_1 z - N^2\sinh\Delta_2 z\right)\times \\ &\times\left((2N^2-k_1^2)\sinh\Delta_1 h - 2\Delta_1\Delta_2\sinh\Delta_2 h\right)\big], \end{aligned} \tag{12}$$

$$\widetilde{\Delta} = 4N^2\Delta_1\Delta_2(2N^2-k_1^2) - (8N^4 - 4N^2k_1^2 + k_1^4)\Delta_1\Delta_2\cosh\Delta_1 k\cosh\Delta_2 k +$$

$$+ N^2(8N^4 - 4N^2k_1^2\frac{3\kappa+1}{\kappa+1} + k_1^4\frac{5\kappa-3}{\kappa+1})\sinh\Delta_1 k\sinh\Delta_2 k.$$

Based on the formulas (6), (7), the transformants of the remaining displacement were found

$$U_{\alpha\beta}(z;k_1,k_2) = \frac{\alpha}{N^2}Z_{\alpha\beta}(z;k_1,k_2),\ V_{\alpha\beta}(z;k_1,k_2) = \frac{i\beta}{N^2}Z_{\alpha\beta}(z;k_1,k_2).$$

Thus, an exact solution of the vector problem (9) (10) in the space of transformants was obtained.

4. Construction of original solutions.

Let's introduce functions dependent on N

$$F_W(N,z;k_1,k_2) = \big[\big(\Delta_1\Delta_2\sinh\Delta_2 z - N^2\sinh\Delta_1 z\big)\times \\ \times\big(2N^2\cosh\Delta_2 h - (2N^2-k_1^2)\cosh\Delta_1 h\big) + N^2(\cosh\Delta_2 z - \cosh\Delta_1 z)\times \\ \times\big((2N^2-k_1^2)\sinh\Delta_1 h - 2\Delta_1\Delta_2\sinh\Delta_2 h\big)\big],$$

$$F_Z(N,z;k_1,k_2) = [\Delta_1\Delta_2(\cosh\Delta_1 z - \cosh\Delta_2 z)\times \\ \times\big(2N^2\cosh\Delta_2 h - (2N^2-k_1^2)\cosh\Delta_1 h\big) + \big(\Delta_1\Delta_2\sinh\Delta_1 z - N^2\sinh\Delta_2 z\big)\times \\ \times\big((2N^2-k_1^2)\sinh\Delta_1 h - 2\Delta_1\Delta_2\sinh\Delta_2 h\big)\big].$$

After applying inverse integral transformations to the solution of (12), the original displacements were obtained

$$W(x,y,z;k_1,k_2) = -\frac{P}{G\pi^2}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{\Delta_2}{\widetilde{\Delta}}F_W(N,z)\cos\alpha a\, e^{-i\beta(y-b)}\cos\alpha x\, d\beta\, d\alpha,$$

$$V(x,y,z;k_1,k_2) = -\frac{P}{G\pi^2}\frac{\partial}{\partial y}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{N^2}{\widetilde{\Delta}}F_Z(N,z)\cos\alpha a\, e^{-i\beta(y-b)}\cos\alpha x\, d\beta\, d\alpha,$$

$$U(x,y,z;k_1,k_2) = -\frac{P}{G\pi^2}\frac{\partial}{\partial x}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{N^2}{\widetilde{\Delta}}F_Z(N,z)\cos\alpha a\, e^{-i\beta(y-b)}\cos\alpha x\, d\beta\, d\alpha.$$

Using the parity of the function related to the variable α under the integral and applying Euler's formula, the displacements are rewritten in the form

$$W(x,y,z;k_1,k_2) = -\frac{P}{4G\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\Delta_2}{\widetilde{\Delta}}F_W(N,z)e^{-i\alpha(a-x)-i\beta(y-b)}d\beta\, d\alpha,$$

$$V(x,y,z;k_1,k_2) = -\frac{P}{4G\pi^2}\frac{\partial}{\partial y}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{N^2}{\widetilde{\Delta}}F_Z(N,z)e^{-i\alpha(a-x)-i\beta(y-b)}d\beta\, d\alpha,$$

$$U(x,y,z;k_1,k_2) = -\frac{P}{4G\pi^2}\frac{\partial}{\partial x}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{N^2}{\widetilde{\Delta}}F_Z(N,z)e^{-i\alpha(a-x)-i\beta(y-b)}d\beta\, d\alpha.$$

In order to get rid of the double integral by the parameters of the Fourier transforms, the relation connecting the Fourier and Hankel transforms was used [13]

$$\frac{1}{4\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}F\left(\sqrt{\alpha^2+\beta^2+\chi_i^2}\right)e^{-i\alpha x-i\beta y}d\alpha d\beta = \int\limits_{0}^{\infty}sF(\sqrt{s^2+\chi_i^2})\times$$
$$\times J_0(s\sqrt{x^2+y^2})ds,$$

where $J_0(s)$ is the Bessel function, $\chi_1 = k_1,\ \chi_2 = k_2$. After simplification, the displacement formula takes the form

$$W(x,y,z;k_1,k_2) = -\frac{P}{\pi G}\int\limits_{0}^{\infty}\frac{F_W(s,z)}{\Delta_s}\cdot s\left[J_0(s\sqrt{(x-a)^2+(y-b)^2})+\right.$$
$$\left.+J_0(s\sqrt{(x+a)^2+(y-b)^2})\right]ds,$$

$$V(x,y,z;k_1,k_2) = -\frac{P}{\pi G}\frac{\partial}{\partial y}\int\limits_{0}^{\infty}\frac{F_Z(s,z)}{\Delta_s}\cdot s\left[J_0(s\sqrt{(x-a)^2+(y-b)^2})+\right.$$
$$\left.+J_0(s\sqrt{(x+a)^2+(y-b)^2})\right]ds,$$

$$U(x,y,z;k_1,k_2) = -\frac{P}{\pi G}\frac{\partial}{\partial x}\int\limits_{0}^{\infty}\frac{F_Z(s,z)}{\Delta_s}\cdot s\left[J_0(s\sqrt{(x-a)^2+(y-b)^2})+\right.$$
$$\left.+J_0(s\sqrt{(x+a)^2+(y-b)^2})\right]ds,$$

$$F_W(s,z) = \delta_2\left[\left(\delta_1\delta_2\sinh\delta_2 z - s^2\sinh\delta_1 z\right)\times\right.$$

$$\times\left(2s^2\cosh\delta_2 h-(2s^2-k_1^2)\cosh\delta_1 h\right)+$$
$$+\,s^2\left(\cosh\delta_2 z-\cosh\delta_1 z\right)\left((2s^2-k_1^2)\sinh\delta_1 h-2\delta_1\delta_2\sinh\delta_2 h\right)\big],$$

$$F_Z(s,z)=N^2\left[\delta_1\delta_2\left(\cosh\delta_1 z-\cosh\delta_2 z\right)\times\right.$$
$$\times\left(2s^2\cosh\delta_2 h-(2s^2-k_1^2)\cosh\delta_1 h\right)+$$
$$\left.+\left(\delta_1\delta_2\sinh\delta_1 z-s^2\sinh\delta_2 z\right)\left((2s^2-k_1^2)\sinh\delta_1 h-2\delta_1\delta_2\sinh\delta_2 h\right)\right],$$

$$\widetilde{\Delta}_s=4s^2\delta_1\delta_2(2s^2-k_1^2)-(8s^4-4s^2k_1^2+k_1^4)\delta_1\delta_2\cosh\delta_1 h\cosh\delta_2 h+$$
$$+\,s^2\left(8s^4-4s^2k_1^2\frac{3\kappa+1}{\kappa+1}+k_1^4\frac{5\kappa-3}{\kappa+1}\right)\sinh\delta_1 h\sinh\delta_2 h,$$

where $\delta_1=\sqrt{s^2-k_1^2},\ \delta_2=\sqrt{s^2-k_2^2}$.

Using the parity of the Bessel function $J_0(s)$, we will continue the integration in an odd way to the interval $(-\infty,\ 0)$, we will find the displacement from the load distributed over a rectangular area

$$W^{AB}(x,y,z;k_1,k_2)=-\frac{P}{\pi G}\int\limits_0^A\int\limits_{-B}^B\int\limits_{-\infty}^{\infty}\frac{F_W(s,z)}{\Delta_s}\cdot s\times$$
$$\times\left[J_0(s\sqrt{(x-a)^2+(y-b)^2})+J_0(s\sqrt{(x+a)^2+(y-b)^2}\right]ds\,da\,db,$$

$$V^{AB}(x,y,z;k_1,k_2)=-\frac{P}{\pi G}\frac{\partial}{\partial y}\int\limits_0^A\int\limits_{-B}^B\int\limits_{-\infty}^{\infty}\frac{F_Z(s,z)}{\Delta_s}\cdot s\times$$
$$\times\left[J_0(s\sqrt{(x-a)^2+(y-b)^2})+J_0(s\sqrt{(x+a)^2+(y-b)^2}\right]ds\,da\,db,$$

$$U^{AB}(x,y,z;k_1,k_2)=-\frac{P}{\pi G}\frac{\partial}{\partial x}\int\limits_0^A\int\limits_{-B}^B\int\limits_{-\infty}^{\infty}\frac{F_Z(s,z)}{\Delta_s}\cdot s\times$$
$$\times\left[J_0(s\sqrt{(x-a)^2+(y-b)^2})+J_0(s\sqrt{(x+a)^2+(y-b)^2}\right]ds\,da\,db.$$

Using the results of the works [14], [3] and integral representation of the Bessel function, on the transformation of the integral, write the displacements in the forms

$$W^{AB}(x,y,z;k_1,k_2) = -\frac{4P}{\pi GN}\int\limits_0^\infty \frac{F_W(s,z)}{\Delta_s}\times$$
$$\sum_{k=1}^{N}\frac{\cos sx\sqrt{1-\tau_k^2}\sin sA\sqrt{1-\tau_k^2}\cos sy\tau_k \sin sB\tau_k}{s\tau_k\sqrt{1-\tau_k^2}}ds, \quad (13)$$

$$V^{AB}(x,y,z;k_1,k_2) = \frac{4P}{\pi GN}\int\limits_0^\infty \frac{F_Z(s,z)}{\Delta_s}\times$$
$$\sum_{k=1}^{N}\frac{\cos sx\sqrt{1-\tau_k^2}\sin sA\sqrt{1-\tau_k^2}\sin sy\tau_k \sin sB\tau_k}{s\sqrt{1-\tau_k^2}}ds,$$

$$U^{AB}(x,y,z;k_1,k_2) = \frac{4P}{\pi GN}\int\limits_0^\infty \frac{F_Z(s,z)}{\Delta_s}\times$$
$$\sum_{k=1}^{N}\frac{\sin sx\tau_k \sin sA\tau_k \cos sy\sqrt{1-\tau_k^2}\sin sB\tau_k}{s\sqrt{1-\tau_k^2}}ds,$$

where $\tau_k = \cos\left(\frac{2k-1}{2N}\pi\right)$ — zeros of the Chebyshev polynomial of the 1st kind.

5. Results of numerical calculations. The graphs represented below are distribution for vertical displacement on the upper face $W^{AB}(x,y,h;k_1,k_2)$ from (13) for the values of Poisson's ratio $\mu = \frac{1}{3}$ and $\mu = \frac{1}{4}$ for frequencies, using formulas (8) $\omega = 0.3;\ 1;\ 3,\ \rho = 8.5,\ G = 40,\ h = 1.$ Three forms of the load distribution section along the face $z = h$ are considered

1. $B = A/2$ the load is distributed across the square;
2. $B = A$ - the load is distributed along a rectangle stretched along the Oy axis;
3. $B = A/4$ - the load is distributed over a rectangle stretched along the Ox axis.

Comparing the values of vertical displacements for different values of Poisson's ratio, it can be seen that the behavior of the graph is similar, but for values

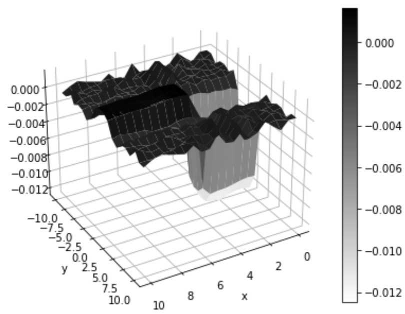

Fig. 1. $B = A/2,\ \omega = 0.3,\ \mu = 1/3$

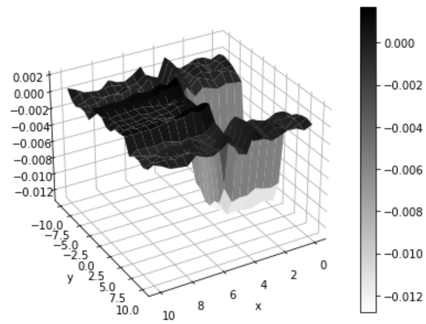

Fig. 2. $B = A,\ \omega = 1,\ \mu = 1/3$

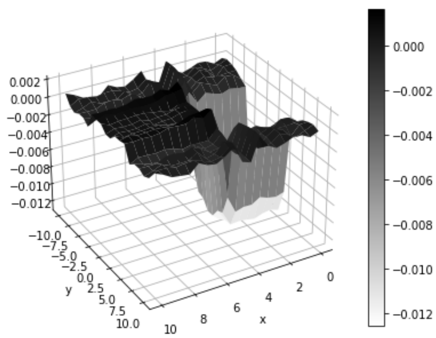

Fig. 3. $B = A,\ \omega = 0.3,\ \mu = 1/3$

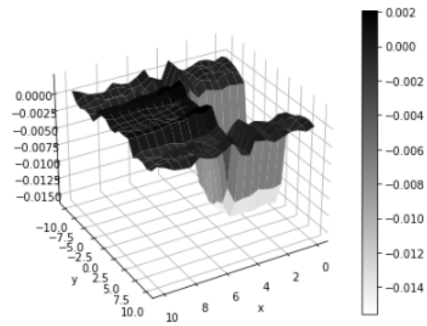

Fig. 4. $B = A,\ \omega = 3,\ \mu = 1/3$

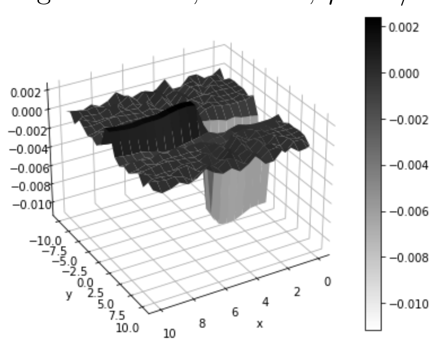

Fig. 5. $B = A/4,\ \omega = 0.3,\ \mu = 1/3$

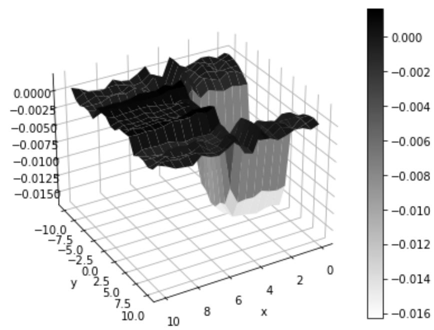

Fig. 6. $B = A,\ \omega = 0.3,\ \mu = 1/4$

$\mu = 1/4$ the amplitude of oscillations is larger (Fig. 3, Fig. 6)). Comparing the graphs of vertical displacements for the same frequency $\omega = 0.3$ and Poisson's ratio $\mu = 1/3$ under different sections of the load distribution (Fig. 1, Fig. 3, Fig. 5), it can be seen that the maximum absolute values achieved with the shape of the section $B = A$, which corresponds to a rectangle elongated along the y-axis. In the case when the load is distributed over a rectangle elongated along the x-axis, the displacement has a minimum amplitude and its maximum displacement is about -0.01 units (Fig. 5). In the case when the load is distributed over the rectangle $B = A$, with an increase in the vibration frequency (Fig. 2, Fig. 3, Fig. 4), the amplitude of displacement grows. Positive displacements are observed, which means the lifting of the face of the elastic layer. The maximum absolute values achieved with $\omega = 3$ (Fig. 4).

Conclusion

The dynamical problem's solution of the elasticity for the elastic layer was derived, when the lower face of the layer is rigidly fixed to the foundation, the side border is in the smooth contact, and upper face is under the influence of the normal dynamic compressive load, applied at the initial moment of time and distributed across a rectangular section. Application of the integral transform method directly to the movement equations reduced the initial problem to the one-dimensional vector problem. The last one was solved exactly using the matrix differential calculus. The proposed approach makes it possible to obtain an exact solution of the problem in the transform's space.

In the future, it is possible to consider different cases of boundary conditions and evaluate the influence of the defect inside the layer on displacements and stresses.

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Динамічна задача про дію розподіленого навантаження на пружний шар

Резюме

Побудовано хвильове поле пружного півшару, коли на одній грані у початковий момент часу діє динамічне нормальне навантаження, розподілене за прямокутною ділянкою. Нижня границя півшару жорстко зчеплена з основою, а торець знаходиться в умовах гладкого контакту. Використовується метод розвалу системи рівнянь руху на систему рівнянь та незалежно розв'язуване рівняння, цей підхід був запропонований Поповим Г. Я. Застосовуються інтегральні перетворення Лапласа та Фур'є безпосередньо до рівнянь руху та крайових умов, що зводить задачу до векторної одновимірної крайової задачі, яку розв'язано методом матричного диференційного числення. Вихідні переміщення отримано застосуванням обернених інтегральних перетворень. Розглянуто випадок усталених коливань та проаналізовано амплітуду вертикальних переміщень, що виникають у шарі в залежності від форми ділянки розподіленого навантаження, матеріалу середовища шару та значень власної частоти коливань шару.

Ключові слова: точний розв'язок, динамічне навантаження, пружний шар, інтегральні перетворення.

References

1. Batra, R. C., Jiang, W. (2008). Analytical solution of the contact problem of a rigid indenter and an anisotropic linear elastic layer. *International Journal of Solids and Structures,* Vol. 45, №. 22-23, P. 5814–5830.
2. Fesenko, A. A. (2019). An exact solution of the dynamical problem for the infinite elastic layer with a cylindrical cavity. *Researches in Mathematics and Mechanics,* Vol. 24, №2(34), P. 75–87.
3. Fesenko, A. A., Bondarenko, K. S.(2020). The dynamical problem on acting concentrated load on the elastic quarter space. *Researches in Mathematics and Mechanics,* Vol. 25, №2 (36), P. 7–26.
4. Grinchenko, V. T. (1981). *Harmonicheskie kolebaniya i volny v uprugih telah [Harmonic vibrations and waves in elastic bodies]* Kiev: Naukova Dumka, p. 284.
5. Kuznetsova, E. L., Tarlakovskii, D. V. and Fedotenkov, G. V. (2011). Propagation of unsteady waves in an elastic layer. *Mechanics of solids,* Vol. 46, P. 779–787.
6. Miroshnikov V. Y. (2020). Stress state of an elastic layer with a cylindrical cavity on a rigid foundation *International Applied Mechanics,* Vol. 56, №3, P. 372–381.
7. Popov, G. Ya. (2002). O privedenii uravneniy dvizeniya uprugoy sredy k odnomu nezavisimomu i k dvum sovmestno reshaemym uravneniyam [On reducing the equations of motion of an elastic medium to one independent and to two jointly solvable equations]. *DAN,* Vol. 384(2), P. 193–196.
8. Popov, G. Ya., (2003). Tochnoe reshenie smeshannoy zadachi teorii uprugosti dlya chetvert'prostranstva [An exact solution to the mixed problem of the elasticity theory for a quarter-space]. *Izvestia RUN. Rigid body mechanics,* Vol. 6, P. 31–39.
9. Popov, G. Ya., Abdimanapov, S. A. and Efimov, V. V. (1999). *Fuctii i matriti Greena odnomernyh kraevah zadach [Green's functions and matrix of onedimensional boundary value problems.]* Almati: Rauan, 113 p.
10. Popov, G. Ya., Vaysfeld, N. D. (2010). Ob odnom podhode k resheniyu zadachi Lamba [One new approach to solving the Lamb problem]. *Doclady RUN,* Vol. 432, №3, P. 337–342.
11. Poruchnikov, V. B. (1986). *Methods of dynamic theory of elasticity*, Moscow: Nauka, p. 328.
12. Rabinovich, A. S. (1974). Plane contact problem on the pressure of a stamp with a rectangular base on a rough elastic half-space. *Izv. Akad. Nauk. ArmSSR, Mekhanika.* Vol. 27, №4.
13. Sneddon, I. N. (1955).*Preobrazovanie Furie [Fourier transforms]*, Moscow: Izdat. Inostr. Lit., p. 667.

14. Sonin, N. Y. (1954). *Issledovanie tsilindricheskih functiy i specialnih polinomov [Research on cylindrical functions and special polynomials]*, Moscow: Gostechizdat.

15. Vaysfeld, N. & Fesenko, A. (2019). *Zmishani zadachi teoriy pruzhnosti dlya pivneskinchennogo sharu [The mixed partial elasticity problems for a semi-infinite layer]*, Odesa: Astroprint, p. 120.

16. Winfried Schepers, Stavros Savidis and Eduardo Kausel. (2010). *Dynamic stresses in an elastic half-space*, SoilDynamics and Earthquake Engineering, Vol. 30(9), P. 833–843.