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# MODELING OF DEFORMATION OF THE BIMATERIAL WITH THIN NON-LINEAR INTERFACE INCLUSION

An incremental approach to solving the antiplane problem for bimaterial media with a thin, physically nonlinear inclusion placed on the materials interface is discussed. Using the jump functions method and the coupling problem of boundary values of the analytical functions method we reduce the problem to the system of singular integral equations (SSIE) on jump functions with variable coefficients allowing us to describe any quasi-static loads (monotonous or not) and their influence on the stress-strain state in the bulk. To solve the SSIE problem, an iterative analytical-numerical method is offered for various non-linear deformation models. Numerical calculations are carried out for different values of non-linearity characteristic parameters for the inclusion material. Their parameters are analyzed for a deformed body under a load of a balanced concentrated force system.

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## 1. Introduction

Problems related to contacts have been treated with great attention in the literature of the subject due to their practical significance. Most materials contain numerous subtle defects in the form of cracks and inclusions of various origin [4; 7; 10; 13; 14; 18; 20; 23]. The presence of these inhomogeneities in engineering materials affects or disturbs their elastic field and thus greatly influences their mechanical and physical properties. Composite materials take advantage of inclusions as reinforcements in the matrix to have superior properties not achievable by individual constituent materials. Such subtle inhomogeneities can have a complex structure, taking into account possible viscosity, plasticity, and other nonlinear effects. Considering non-linearity significantly

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complicates the process of solving the problem and requires the use of various approximate methods even for bodies of simple geometry [1; 5; 12; 18; 19].

It has been noted in the surveys [13–15; 22; 23] that for solving the problem for elastic bodies with thin inclusions it is possible to select five main approaches of analysis: **general theoretic** — to consider the inclusion of arbitrary form and then to decrease one of its sizes [6; 13–15]; **numerical** — to apply direct numerical methods [16]; **experimental** — to use experimental methods; **asymptotical** — to consider the stresses and displacements directly near the vicinity of heterogeneity and interface of materials by asymptotical methods in detail; **new theories of imperfect contact** — to develop a specific theory that would enable to solve the proper problems rather simple taking into account the effect of the small thickness of the defect [8; 9; 18; 23].

The idea of the last, one of the most productive approaches, is based on the principle of the conjugation of continua with different dimensions [18; 23]. An object is eliminated from consideration and its influence results in the appearance of jumps of temperature, heat fluxes, vectors of displacements and stresses in the matrix. Then stresses and other characteristics in an arbitrary point of solid are determined by the problem geometry, materials properties, external loading and jump functions. The mathematical model of inclusion is given as the **interaction conditions** equivalent to the conditions of imperfect contact between the matrix surfaces adjacent to inclusion.

Attempts to consider non-linearity in the problem of antiplane deformation of compressed semi-spaces with thin interfacial defects were made by various authors, including the study of sliding friction of contact bodies [3; 4; 7; 9; 21].

This article aims to develop a jump functions method and construct appropriate models of thin inclusions and layers whose material has essentially non-linear properties. Assume that the body thus loads non-uniformly, including multistage or cyclic loads.

## Main Results

**1. Formulation of the problem.** Consider an infinite isotropic matrix consisting of two semi-spaces with the elastic shear constants $G_k$ $(k = 1, 2)$. Here $Oxyz$ are the Cartesian coordinates and $xOz$ is the plane of contact between half-spaces.

We'll study the stress-strain state (SSS) of the bulk section by the plane

$xOy$ perpendicular to the direction $z$ of its longitudinal shear. This section creates two half-planes $S_k$ $(k = 1, 2)$ and the interface between them corresponds to the $x$-axis $L$ (Fig. 1). On $L$ along the segment $L' = [-a; a]$ there is a thin inclusion of thickness $2h \ll a$, mechanical properties of which in different directions may differ (orthotropy) and are characterized by constitutive equality of rather general non-linear form

$$\frac{\partial w^{in}}{\partial s} = \varpi_s \left(\sigma_{xz}^{in}, \sigma_{yz}^{in}\right), \quad s = \{x, y\}, \tag{1}$$

where the monotone function $\varpi_s \left(\sigma_{xz}^{in}, \sigma_{yz}^{in}\right)$ is chosen from general theoretical considerations or is some kind of approximation of empirical data relationships.

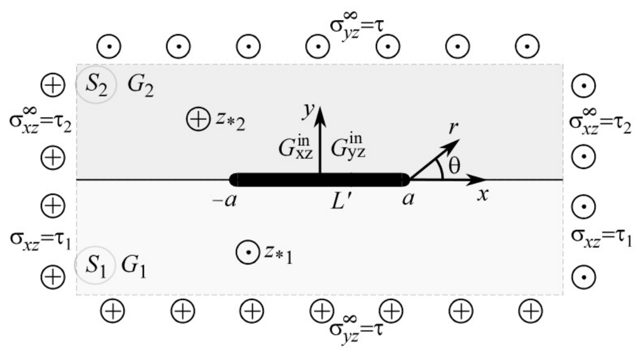


Figure 1: Geometry and load pattern of the problem

Suppose that external loading increase or decrease monotonically by arbitrary law and consist of the uniformly distributed in infinity shear stresses $\sigma_{yz}^{\infty} = \tau(t)$, $\sigma_{xz}^{\infty} = \tau_k(t)$, concentrated forces with magnitude $Q_k(t)$, screw dislocations with Burger's vector $b_k(t)$ in points $z_{*k} \in S_k$ $(k = 1, 2)$, $t$ denotes the time as formally monotonically increasing parameter associated with load variations $(k = 1, 2)$. It should be noted that the positive direction of the forces and Burgers vectors is chosen along the axis $z$, in contrast to the opposite direction implicitly adopted in some studies. Since we assume the straightness of the matrix interface at infinity, we have to provide a correlation $\tau_2(t)G_1 = \tau_1(t)G_2$.

The presence of a thin inclusion in the bulk at the interface of the material is simulated by the jumps of stress tensor components and vectors of displacements on $L'$ [18; 23]

$$\begin{aligned}
&[\sigma_{yz}]_h \cong \sigma_{yz}^- - \sigma_{yz}^+ = f_3(x,t),\\
&\left[\frac{\partial w}{\partial x}\right]_h \cong \frac{\partial w^-}{\partial x} - \frac{\partial w^+}{\partial x} = \left[\frac{\sigma_{xz}}{G}\right]_h \equiv \frac{\sigma_{xz}^-}{G_1} - \frac{\sigma_{xz}^+}{G_2} = f_6(x,t), \quad x \in L';\\
&f_3(x,t) = f_6(x,t) = 0, \quad x \notin L'.
\end{aligned} \tag{2}$$

Hereinafter the following notation is used: $[\varphi]_h = \varphi(x,-h) - \varphi(x,+h)$; $\langle\varphi\rangle_h = \varphi(x,-h) + \varphi(x,+h)$; the "+" and "−" indicators correspond to the limit values of functions at the upper and lower faces of the line $L'$. The contact between the upper and lower faces of inclusion and semi-spaces along the line $L'$ and between semi-spaces along the line $L'' = L \backslash L'$ considered to be mechanically perfect

$$\begin{aligned}
&w^{in}(x,\pm h) = w_k(x,\pm h), \quad \sigma_{yz}^{in}(x,\pm h) = \sigma_{yzk}(x,\pm h) \quad (x \in L'),\\
&w_1(x,-0) = w_2(x,+0), \quad \sigma_{yz1}(x,-0) = \sigma_{yz2}(x,+0) \quad (x \in L'').
\end{aligned} \tag{3}$$

In this way, we formulate the problem of longitudinal shear in the non-bounded matrix with possible non-linear deformation of the thin interface inclusion-layer under the action of the inhomogeneous distribution of shear stresses, concentrated forces and screw dislocations. These forces can cause energy dissipation, wear, etc. in the matrix with inclusion.

**2. Modeling of the presence of thin interface inclusion.** The mathematical model of thin inclusion is presented in the form of so-called interaction conditions [21–23], which are equivalent to the conditions of nonperfect contact between the adjacent inclusion surfaces of the matrix. The basis of the proposed method of simulation of a thin object is the principle of volumetric integration, which consists of defining relationships describing the physical and mechanical state of the inclusion material and then considering the smallness of one of the linear dimensions of the inclusion. The main constitutive relations for the arbitrary material of inclusion are the equilibrium conditions

$$\frac{\partial \sigma_{xz}^{in}}{\partial x} + \frac{\partial \sigma_{yz}^{in}}{\partial y} = 0 \tag{4}$$

and some known stress-strain dependencies (1).

Using thin-walled proportions

$$\frac{\partial w^{in}}{\partial y}(x,h) + \frac{\partial w^{in}}{\partial y}(x,-h) \simeq \frac{w^{in}(x,h) - w^{in}(x,-h)}{h} = -\frac{\left[w^{in}\right]_h}{h}; \tag{5}$$

integrating (4) by $x$ within $[-a, x]$ and averaging in thickness $y \in [-h, h]$

$$\frac{1}{2h}\int_{-h}^{h}\sigma_{xz}^{in}(\xi, y, t)dy \simeq \sigma_{xz}^{inAver}(x,t) = \frac{1}{2}\left\langle\sigma_{xz}^{in}\right\rangle_h(x,t) \tag{6}$$

we obtain the following form of inclusion balance conditions:

$$\frac{1}{2h}\int_{-h}^{h}\sigma_{xz}^{in}(\xi, y, t)dy - \sigma_{xz}^{in}(-a,t) - \frac{1}{2h}\int_{-a}^{x}\left[\sigma_{yz}^{in}\right]_h(\xi,t)d\xi = 0, \tag{7}$$

which, together with the relations (4), (5), fully describe the model of the thin physically non-linear inclusion, presented in SSS inclusion values. To pass to the values of the SSS of the matrix we must use the contact conditions (3). Finally, from (5), (7), and dependence (4), adding (3), we obtained the mathematical model of a thin physically non-linear inclusion [18; 23]

$$\begin{cases} -\dfrac{[w]_h}{h} = \left\langle \varpi_s\left(\sigma_{xz}, \sigma_{yz}, t\right)\right\rangle, \\ \left\langle \varpi_x^{-1}\left(\dfrac{\sigma_{xz}}{G_k}, \dfrac{\sigma_{yz}}{G_k}\right)\right\rangle_h - 2\sigma_{xz}^{in}\left(-a,t\right) - \dfrac{1}{h}\displaystyle\int_{-a}^{x}\left[\sigma_{yz}\right]_h(\xi,t)d\xi = 0. \end{cases} \tag{8}$$

The next step is to clarify the form of relationship (1). The relationship $\varpi_s(\sigma_{xz}^{in}, \sigma_{yz}^{in})$ can be given as an analytical function that reflects a specific deformation graph over the entire quasi-static load range, as well as approximate relationships based on experimental data on the measurement of the deformation properties of specific materials. It should also be noted that dependencies (1) for loading and unloading are predominantly (in the absence of ideal even non-linear elasticity) in a different form [2].

Let's consider a few partial cases of (1):

1. Let $\varpi_s\left(\sigma_{xz}^{in}, \sigma_{yz}^{in}, t\right) = 0$, $s = \{x, y\}$. This option corresponds to a completely rigid inclusion.

2. $\varpi_s\left(\sigma_{xz}^{in}, \sigma_{yz}^{in}, t\right) = const$, $s = \{x, y\}$. This option can be considered a model of ideal hard-plastic deformation of inclusion.

3. Many variants of the dependence form (1) are related to the assumption that the inclusion material is orthotropically non-linear and relation (1) can be written in a simpler form

$$\frac{\partial w^{in}}{\partial x} = \Im_x^{in}\left(\sigma_{xz}^{in}, t\right), \quad \frac{\partial w^{in}}{\partial y} = \Im_y^{in}\left(\sigma_{yz}^{in}, t\right), \tag{9}$$

or

$$\sigma_{xz}^{in} = G_x^{in}\left(\sigma_{xz}^{in}, t\right)\frac{\partial w^{in}}{\partial x}, \quad \sigma_{yz}^{in} = G_y^{in}\left(\sigma_{yz}^{in}, t\right)\frac{\partial w^{in}}{\partial y} \tag{10}$$

with variable shear modulus $G_x^{in}(\sigma_{xz}^{in})$, $G_y^{in}(\sigma_{yz}^{in})$ as specified. For example:

- Hooke's classic linear law of elasticity

$$\sigma_{xz}^{in} = G_x^{in}\frac{\partial w^{in}}{\partial x}, \quad \sigma_{yz}^{in} = G_y^{in}\frac{\partial w^{in}}{\partial y}. \tag{11}$$

- Bach-Shule plastic deformation model $\Im_s^{in}(\sigma_{sz}^{in}) = K(\sigma_{sz}^{in})^{\beta}$.
- Ramberg-Osgood deformation model,

$$\frac{\partial w^{in}}{\partial s} = A_s\sigma_{sz}^{in}\left(1 + B_s\left(\sigma_{sz}^{in}\right)^{M_s}\right), \quad s = \{x, y\}, \tag{12}$$

  where the ratio (12) coincides with the deformation model variant [2] in the case of $M_s = m_s - 1$, $A_s = 1/G_{os}$, $B_s = K_s A_s^{m_s-1}$, $s = \{x; y\}$ ($G_{os}$, $m_s$, $K_s$ are inclusion material parameters) and when $K_s = 0.002\left(\frac{G_{0s}}{G_{0.2T}}\right)^{m_s}$, $s = \{x, y\}$ it is used to determine the "technical yield point" and the particular material parameter $m_s$. We note that model (12) can be considered as a non-linear ideal elastic deformation option.

- Ilyushin's model of plastic deformation

$$\Im_s^{in}(\sigma_{sz}^{in}) = \frac{\sigma_{sz}^{in}}{G_{0s}^{in}\left(1 - \omega(\sigma_{sz}^{in})\right)}. \tag{13}$$

- Classic linear elastic-plastic deformation models without hardening

$$\begin{cases} \dfrac{\partial w^{in}}{\partial s} = \dfrac{\sigma_{sz}^{in}}{G_{0s}}, & \left|\sigma_{sz}^{in}\right| < \tau_{yield}, \quad s = \{x, y\}, \\ \dfrac{\partial w^{in}}{\partial s} = \dfrac{\tau_{yield}}{G_{0s}}, & \left|\sigma_{sz}^{in}\right| \geq \tau_{yield} \end{cases} \tag{14}$$

  and with hardening

$$\begin{cases} \dfrac{\partial w^{in}}{\partial s} = \dfrac{\sigma_{sz}^{in}}{G_{0s}}, & \left|\sigma_{sz}^{in}\right| < \tau_{yield}, \quad s = \{x, y\}, \\ \dfrac{\partial w^{in}}{\partial s} = \left(\sigma_{sz}^{in} - \tau_{yield}\right)\dfrac{G_{0s} - G_{1s}}{G_{0s}G_{1s}}, & \left|\sigma_{sz}^{in}\right| \geq \tau_{yield}. \end{cases} \tag{15}$$

- Models of plastic deformation

$$\left(\Im_s^{in}(\sigma_{sz}^{in})\right)^{-1} = \frac{k\varepsilon_{sz}^{in}}{\sqrt{1+(\varepsilon_{sz}^{in}/m)^2}};$$
$$\left(\Im_s^{in}(\sigma_{sz}^{in})\right)^{-1} = \tau_{yield} + \frac{G_{sT}\varepsilon_{sz}^{yield}}{1-G_{sT}/G_{s0}}. \tag{16}$$

- Each deformation model is given by a function like the experimentally obtained dependence $\Im_s^{in}\left(\sigma_{sz}^{in},t\right)$ for a given material [2].

For any of the above-mentioned deformation models of the form (9), considering (10) and (3), the mathematical model of physically orthotropically non-linear thin inclusion will take the following form

$$\begin{cases} G_x^{in}(\sigma_{xz}^{in},t)\left\langle \dfrac{\partial w}{\partial x}\right\rangle_h (x,t) - 2\sigma_{xz}^{in}(-a,t) - \dfrac{1}{h}\displaystyle\int_{-a}^{x}\left[\sigma_{yz}\right]_h(\xi,t)d\xi = 0, \\ G_y^{in}(\sigma_{yz}^{in},t)\,[w]_h(x,t) + h\left\langle\sigma_{yz}\right\rangle_h(x,t), = 0. \end{cases} \tag{17}$$

Of course, the model (17) can be further complicated by considering the more complicated dependence (1), the nonperfect contact between the matrix and inclusion [8; 11; 19], the thermal load, etc. However, these complications will not be of fundamental importance for the general methodology of solving the problem.

**3. The problem solution.** Using the method [18; 23] to solve the problem, we can obtain dependences of stress tensor components and vector displacement derivatives on the line $L$ of the unbounded plane at the load stage

$$\begin{aligned}
&\sigma_{yz}^{\pm}(x,t) = \mp p_k f_3(x,t) - Cg_6(x,t) + \sigma_{yz}^{0\pm}(x,t),\\
&\sigma_{xz}^{\pm}(x,t) = \mp Cf_6(x,t) + p_k g_3(x,t) + \sigma_{xz}^{0\pm}(x,t),\\
&g_r(z,t) \equiv \frac{1}{\pi}\int_{L'}\frac{f_r(x,t)dx}{x-z}, \quad p = \frac{1}{G_1+G_2}, \quad p_k = pG_k, \quad C = G_{3-k}p_k,\\
&\sigma_{yz}(z,t) + i\sigma_{xz}(z,t) = \sigma_{yz}^0(z,t) + i\sigma_{xz}^0(z,t) + ip_k g_3(z,t) - Cg_6(z,t)\\
&\qquad (z\in S_k;\ r = 3,\,6;\ k = 1,\,2)\,.
\end{aligned} \tag{18}$$

Superscript “+” refers to $k = 2$; “−” — $k = 1$. The upper index “0” means the corresponding values in the solid body without heterogeneity under the same external loading (homogeneous solution). The following entries [21] shall continue to apply:

$$\begin{aligned} &\sigma_{yz}^0(z,t)+i\sigma_{xz}^0(z,t)=\tau(t)+i\left\{\tau_k(t)+D_k(z,t)+\right.\\ &\qquad\left.+\left(p_k-p_j\right)\overline{D}_k(z,t)+2p_kD_j(z,t)\right\}, \\ &D_k(z,t)=-\frac{Q_k(t)+iG_kb_k(t)}{2\pi\left(z-z_{*k}\right)}\quad\left(z\in S_k,\ k=1,2;\ j=3-k\right), \end{aligned} \tag{19}$$

Using (18), (19) and boundary conditions (3) the problem reduces (17) to a system of singular integral equations (SSIE)

$$\begin{cases} (p_2-p_1)f_6(x,t)+2pg_3(x,t)-\dfrac{1}{hG_x^{in}(\sigma_{xz}^{in})}\displaystyle\int_{-a}^{x}f_3(\xi,t)d\xi=F_3\left(x,G_x^{in}(\sigma_{xz}^{in}),t\right), \\ (p_2-p_1)f_3(x,t)+2Cg_6(x,t)-\dfrac{G_y^{in}(\sigma_{yz}^{in})}{h}\displaystyle\int_{-a}^{x}f_6(\xi,t)d\xi=F_6\left(x,G_y^{in}(\sigma_{yz}^{in}),t\right), \end{cases}$$

$$\begin{aligned} &F_3\left(x,G_x^{in}(\sigma_{xz}^{in}),t\right)=\frac{2}{G_x^{in}(\sigma_{xz}^{in})}\sigma_{xz}^{in}(-a)-\left(\sigma_{xz2}^0(x,t)/G_2+\sigma_{xz1}^0(x,t)/G_1\right), \\ &F_6\left(x,G_y^{in}(\sigma_{yz}^{in}),t\right)=\left\langle\sigma_{yz}^0\right\rangle(x,t)-G_y^{in}(\sigma_{yz}^{in})\left(\frac{\sigma_{yz2}^0(x,t)}{G_2}+\frac{\sigma_{yz1}^0(x,t)}{G_1}\right)- \\ &\qquad-\frac{G_y^{in}(\sigma_{yz}^{in})}{h}\left[w^0\right](-a) \end{aligned} \tag{20}$$

with additional power balance conditions and the uniqueness of the displacements when traversing a thin defect

$$\int_{-a}^{a}f_3(\xi,t)d\xi=2h\left(\sigma_{xz}^{in}(a)-\sigma_{xz}^{in}(-a)\right),\qquad\int_{-a}^{a}f_6(\xi,t)d\xi=[w](a)-[w](-a). \tag{21}$$

The method [21; 23] can be used to solve SSIE (20), (21) because the characteristic part of SSIE does not depend on non-linear coefficients [17].

In general, local jump of displacement and energy dissipation are defined by expressions

$$[w](x,t)=\int_{-a}^{x}f_6(\xi,t)d\xi,\quad x\in L'; \tag{22}$$

$$W^d(t)=\int_{L'}\sigma_{yz}(x,t)\,[w](x,t)dx. \tag{23}$$

In partial cases of crack ($G_x^{in}$, $G_y^{in} \to 0$) or rigid inclusion ($G_x^{in}$, $G_y^{in} \to \infty$), SSIE (20) has analytical solutions that correspond to known results [21; 23]. In case when the materials of the semi-spaces are identical ($G_1 = G_2 = G$) SSIE (20) is simplified to two independent SIE:

$$\begin{aligned} &\frac{1}{G}g_3(x,t) - \frac{1}{hG_x^{in}(\sigma_{xz}^{in},t)}\int_{-a}^{x} f_3(\xi,t)d\xi = F_3\left(x, G_x^{in}(\sigma_{xz}^{in},t),t\right), \\ &Gg_6(x,t) - \frac{G_y^{in}(\sigma_{yz}^{in},t)}{h}\int_{-a}^{x} f_6(\xi,t)d\xi = F_6\left(x, G_y^{in}(\sigma_{yz}^{in},t),t\right); \end{aligned} \tag{24}$$

A more detailed analysis of the solution of the problem will be carried out for the partial case (24) equality of elastic characteristics of semi-spaces. As a result of the above-mentioned method [21; 23], we use the decomposition of jump functions into a series of Chebyshev polynomials

$$f_r\left(\frac{x}{a},t\right) = \frac{a}{\sqrt{a^2-x^2}}\sum_{j=0}^{n} B_j^r(t)T_j\left(\frac{x}{a}\right), \quad (r=3,6). \tag{25}$$

Using known integrals, we obtain

$$\int_{-a}^{x} f_r(\xi,t)d\xi = \left(\frac{\pi}{2} + \arcsin\frac{x}{a}\right)aB_0^r(t) - \sqrt{a^2-x^2}\sum_{j=1}^{n}\frac{1}{j}B_j^r(t)U_{j-1}\left(\frac{x}{a}\right),$$

$$g_r\left(\frac{x}{a},t\right) = \sum_{j=1}^{n} B_j^r(t)U_{n-1}\left(\frac{x}{a}\right), \quad \int_{-a}^{a} f_r(\xi,t)d\xi = \pi a B_0^r(t). \tag{26}$$

Next, after transforming (24) into a dimensionless form, using (25)–(26) in the set of points $x_m = \cos\frac{m\pi}{n+1}$ ($m = \overline{1,n}$) generates two independent linear algebraic equations (SLAE) of orders $n+1$ for unknown items $B_j^r$ ($r = 3,6$; $j = \overline{0,n}$)

$$\left\{\begin{aligned} &\sum_{j=0}^{n}\chi_{mj}^3\left(x_m,\tilde{G}_{xm}^{in}\right)B_j^3 = \tilde{F}_3\left(x_m,\tilde{G}_{xm}^{in}\right), \\ &B_0^3 = 2\tilde{h}\left(\sigma_{xz}^{in}(a) - \sigma_{xz}^{in}(-a)\right)/G_{av}, \end{aligned}\right. \tag{27}$$

$$\left\{\begin{aligned} &\sum_{j=0}^{n}\chi_{mj}^6\left(x_m,\tilde{G}_{ym}^{in}\right)B_j^6 = \tilde{F}_6\left(x_m,\tilde{G}_{ym}^{in}\right), \\ &B_0^6 = [\tilde{w}](a) - [\tilde{w}](-a), \end{aligned}\right. \quad m = \overline{1,n}. \tag{28}$$

where the notations are used

$$\chi^3_{jm} = -\delta_{0j}\frac{\gamma_m}{\tilde{G}^{in}_{xm}} + (1-\delta_{0j})\left(\frac{\mu_{jm}}{\tilde{G}^{in}_{xm}} + 2\tilde{p}\right)\rho_{jm},$$
$$\chi^6_{jm} = -\delta_{0j}\tilde{G}^{in}_{ym}\gamma_m + (1-\delta_{0j})\left(\tilde{G}^{in}_{ym}\mu_{jm} + 2\tilde{C}\right)\rho_{jm},$$
$$\mu_{jm} = \frac{\sqrt{1-x_m^2}}{j\tilde{h}}, \quad \gamma_m = \frac{\pi}{\tilde{h}}\left(1 - \frac{m}{n+1}\right), \quad \rho_{jm} = U_{j-1}(x_m),$$
$$\tilde{G}^{in}_{sm} = \tilde{G}^{in}_{s}(x_m), \quad s = \{x, y\},$$
$$\tilde{x} = x/a, \quad \tilde{h} = h/a, \quad \tilde{y} = y/a, \quad \tilde{G}^{in}_x = G^{in}_x/G_{av}, \quad \tilde{G}^{in}_y = G^{in}_y/G_{av},$$
$$\tilde{f}_3 = G_{av}f_3, \quad \tilde{f}_6 = f_6, \quad \tilde{F}_3 = F_3/G_{av}, \quad \tilde{F}_6 = F_6/G_{av}, \quad \tilde{p} = G_{av}p$$
$$\tilde{C} = C/G_{av}, \quad G_{av} = \left\{\sqrt{G_1G_2}, \max\left(G_1, G_2\right), \tau, \tau_{yield}, Q/\pi a\right\},$$

$\delta_{0j}$ is Kronecker symbol.

Dependence $G^{in}_x(\sigma^{in}_{xz}, t)$, $G^{in}_y(\sigma^{in}_{yz}, t)$ on the current SSS causes serious calculation difficulties due to its variability along $L$. Therefore, to take this effect into account, we can propose the following iterative strategy for solving the problem.

Let's mark $\left(G^{in}_{sm}\left(\sigma^{in}_{sz}(x_m), t\right)\right)^k$, $s = \{x, y\}$ is a dependent shear module at collocation points $x_m$ ($m = \overline{1, n}$) for the appropriate number of approximations $k$. At the initial moment (zero approximation), the values $\left(G^{in}_{sm}\right)^0(0, 0)$ are selected as equal to the initial point of the loading process $G^{in}_{x0}$, $G^{in}_{y0}$ (according to the deformation diagram) in the absence of a residual SSS. These values are the same at all points of the collocation $x_m$ ($m = \overline{1, n}$).

1. The external loading of the body starts with a relatively small value of the parameter $\tau$ or $Q$ for the selected loading scheme (the first loading step in time $t_{(1)}$ is completed). Then we solve the SLAE (27)–(28). The obtained values $\left(B^r_j\right)^k$ ($r = 3, 6$; $j = \overline{0, n}$) are replaced into the relations (25) - (26), and then in (18), calculating the stresses and deformations in each of the collocation points

$$\begin{aligned}\sigma^{in}_{yz}(\tilde{x}, t) &= \left\langle\sigma^0_{yz}(\tilde{x}, t)\right\rangle/G_{av} - (p_2 - p_1)\tilde{f}_3(\tilde{x}, t) - 2\tilde{C}\tilde{g}_6(\tilde{x}, t),\\ \sigma^{in}_{xz}(\tilde{x}, t) &= 2\sigma^{in}_{xz}(-a)/G_{av} + \int_{-1}^{\tilde{x}}\tilde{f}_3(\xi, t)d\xi/\tilde{h}.\end{aligned} \tag{29}$$

$$\left\langle \frac{\partial w}{\partial x} \right\rangle (\tilde{x}, t) = \left\langle \frac{\sigma_{xz}^0}{G_k} \right\rangle (\tilde{x}, t) + (p_2 - p_1)\tilde{f}_6(\tilde{x}, t) + 2\tilde{p}\tilde{g}_3(\tilde{x}, t),$$

$$\left\langle \frac{\partial w}{\partial y} \right\rangle (\tilde{x}, t) = \frac{[w](-a)}{h} - \frac{1}{h}\int_{-a}^{x} \tilde{f}_6(\xi, t)d\xi + \left\langle \frac{\sigma_{yz}^0}{G_k} \right\rangle (\tilde{x}, t).$$

2. Next, we check that dependence (10) is performed at each collocation point with the specified accuracy, i.e. that the value of the module $G_{sm}^{in}(\sigma_{sz}^{in})$ corresponds to the stress $\sigma_{sz}^{in}(x_m)$ or deformation $\frac{\partial w}{\partial s}^{in}(x_m)$ level obtained by the given deformation graph. If the specified accuracy meets the requirements, we determine the current values of the modules $G_{xm}^{in}(\sigma_{xz}^{in}, t)$, $G_{ym}^{in}(\sigma_{yz}^{in}, t)$ for each collocation point and proceed to the next loading step (on item 1). If not, we repeat the calculation by replacing in SLAE (27)–(28) the module values $G_{sm}^{in}\left(\sigma_{sz}^{in}(x_m, (G_{sm}^{in})^{k-1})\right)$ at each collocation point obtained at the previous approximation, thus minimizing the deviation of the calculated module from that specified in (10). The process is convergent. Once sufficient accuracy has been reached, we return to item 1, continuing with the loading. The values obtained in the first (initial) step of the SSS matrix will affect the residuals in the second step (loading or unloading).

Using (5), (18), (25), (26) the expression of energy dissipation at the load stage take a discrete dimensionless form

$$W^d(t) = \int_{L'} \sigma_{yz}(x, t)[w](x, t)dx = 0.5\tilde{h}G_{av}a^2\tilde{W}^d(t),$$

$$\tilde{W}^d(t) = 0.5\sum_{m=1}^{n} \left\langle \tilde{\sigma}_{yz}(x_m, t) \right\rangle [\tilde{w}]\,(x_m). \tag{30}$$

**4. Numerical analysis and discussion.** Numerical analysis of the solution of the problem is made for a partial case of an equality of elastic characteristics of half-spaces ($G_1 = G_2 = G$) under a gradual alternating load with concentrated forces $\tilde{Q} = Q/aG_{av}$ ($\tilde{Q}_2 = -\tilde{Q}_1 = \tilde{Q}$, $\tilde{z}_{*2} = -\tilde{z}_{*1} = i\tilde{d}$) in the global load as a result of the change $\tilde{Q}$: $[0 \div 10.0]$.

Fig. 3 shows a comparison of the results using the elastic-plastic deformation law (14)–(15) (dashed line, Fig. 2) and Hooke's deformation law (13) (solid line, Fig. 2).

According to the acute change like deformation in the plastic inclusion area (Fig. 3), the change in energy dissipation rate depends on the difference

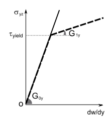

Figure 2: The elastic-plastic deformations law (14)–(15) uniaxial stress-strain diagram for inclusion material

between the diagrams (14)–(15) of the modules $G_{0y}$, $G_{1y}$ (Fig. 4) and is much more pronounced when the points of force application are closer to the inclusion axis.

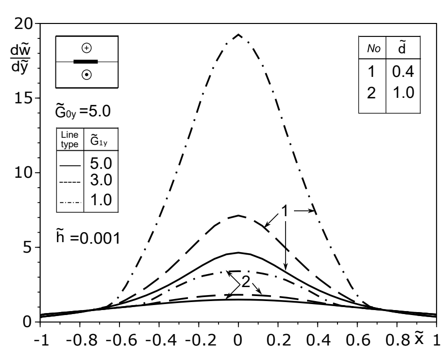


Figure 3: Distribution of elastic-plastic deformations (dash, dot-dash) of the inclusion during loading in comparison with linear elastic (Hooke's law, solid line) deformations

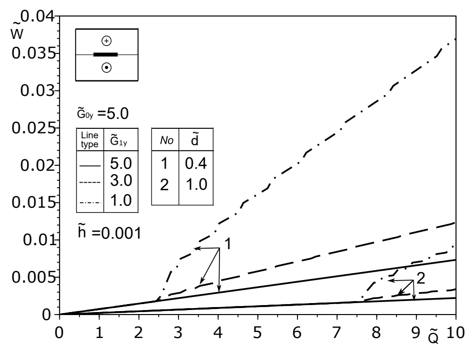


Figure 4: Dependence of change in energy dissipation rate on the difference between the modules in the $G_{0y}$, $G_{1y}$ range (14)–(15) and a distance $\tilde{d}$ of the forces application points from the inclusion axis

## 2. Conclusion

A method of modeling of the thin inclusion with any non-linear physical-mechanical properties of the general form has been developed. This allowed solving the problem of a longitudinal shear of the matrix containing such a thin inclusion-layer at the component's boundary. SLAE with variable coefficients-functions is constructed by the methods of coupling the limit values of analytical functions and jump functions. This makes it possible to describe any way of changing the quasi-static load (monotonous or not) and its effect on the SSS in the body with inhomogeneity based on an incremental approach. For the numerical solution of the system, a convergent iterative analytical-numerical method has been proposed. Calculation formulas for deformations, SSIF, and energy dissipation are constructed. Numerical analysis of the inclusion material is subject to the linear law of elastic-plastic deformation. It has been found that the rate of energy dissipation increased significantly when plastic deformations started under load and are much more pronounced when the points of force application are closer to the inclusion axis.

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*Піскозуб Й. З., Сулим Г. Т.*

Моделювання деформації біматеріалу з тонким механічно нелінійним міжфазним включенням

*Резюме*

Обговорюється інкрементальний підхід до вирішення антиплоскої задачі для біматеріального середовища з тонким, фізично нелінійним міжфазним включенням. Використовуючи методи функцій стрибка та задачі спряження граничних значень аналітичних функцій зводимо задачу до системи сингулярних інтегральних рівнянь (ССІР) зі змінними коефіцієнтами, що дозволяють описувати будь-які квазістатичні навантаження (монотонні і немонотонні), а також їх вплив на напружено-деформований стан середовища. Для вирішення ССІР пропонується ітеративний аналітико-числовий метод для різних моделей нелінійного деформування. Виконуються чисельні розрахунки для різних значень параметрів нелінійності, що характеризують матеріал включення. Їх параметри аналізуються для тіла, що деформується під навантаженням збалансованої системи зосереджених зусиль. Здійснено числові розрахунки для різних значень параметрів нелінійності механічних характеристик матеріалу включення. Досліджено їх вплив на напружено-деформований стан матриці навантаженої збалансованою системою зосереджених сил.

*Ключові слова: нелінійна пружність, тонке включення, розсіювання енергії, коефіцієнт інтенсивності напружень, антиплоскої деформація, поздовжній зсув, біматеріал, функції стрибка.*

*Пискозуб Й. З., Сулим Г. Т.*

Моделирование деформации биматериала с тонким механически нелинейным межфазным включением

*Резюме*

Обсуждается инкрементальный подход к решению антиплоской задачи для биматериальной среды с тонким, физически нелинейным межфазным включением. Используя методы функций скачка и задачи сопряжения граничных значений аналитических функций, мы сводим задачу к системе сингулярных интегральных уравнений (ССИУ) с переменными коэффициентами, позволяющими описывать любые квазистатические нагрузки (монотонные и немонотонные), а также их влияние на напряженно-деформированное состояние среды. Для решения ССИУ предлагается итеративный аналитико-числовой метод для различных моделей нелинейного деформирования. Выполняются численные расчеты для различных значений параметров нелинейности, характеризующих материал включения. Их параметры анализируются для деформируемого тела под нагрузкой сбалансированной системы сосредоточенных усилий. Осуществлено числовые расчеты для различных значений параметров нелинейности упругих характеристик материала включения. Исследовано их влияние на напряженно-деформированное состояние матрицы нагружаемой сбалансированной системой сосредоточенных сил.

*Ключевые слова: нелинейная упругость, тонкое включение, рассеяние энергии, коэффициент интенсивности напряжений, антиплоская деформация, продольный сдвиг, биматериал, функции скачка.*

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