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THE CASE OF ANALYTICAL INVERSION OF LAPLACE TRANSFORM

The new analytical method of inversion of Laplace transforms is proposed in the article for the functions that contain exponents that linearly depend on Laplace transform parameter. This method is based on the transform's expansion into the Taylor series and term-by-term application of Laplace transform inversion. The theorems which confirm the validity and correctness of such approach are proved. This method deals with the generalized functions, so some useful consequences relating with inverse generalized functions are derived. The method is verified by the comparison with the formulas previously known from literature. The new formulas for Laplace transform's originals are given.
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1. Introduction

The integral transforms are widely used in many engineering and mathematical problems. The methods for inversion of Laplace transform are divided into two main groups: analytical and numerical ones. The numerical inversion of Laplace transform causes some doubts for its validity since, as it is well known [1], the Laplace transform inversion problem is not correct one. So, it is important to have new approaches for analytical inversion of Laplace transform despite many developed methods in this area.

The original function can be recovered by the Bromwich contour integral $f(t) = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}ds$ if f is continuous at t [2]. Since the function e^{st} is oscillatory on the contour $(\gamma - i\infty, \gamma + i\infty)$ the approximations of this integral need to know an abscissa of convergence γ. The relations that allow direct calculation of the original function from its transform dispensing contour integration were derived by the change of variables in [3]. The obtained integrals are usually calculated numerically.

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The original function's behavior at the points $t = 0$ and $t \to \infty$ can be found by the initial-value and terminal-value theorems from the transform's function behavior at the points $s \to \infty$ and $s = 0$ respectively if it is known that original functions exist [4], [2]. The asymptotic expansions near some point α_0 can be used $F(s) = \sum\limits_{\nu=0}^{\infty} c_\nu (s - \alpha_0)^{\lambda_\nu}$ if the series is absolutely convergent [5]. In this case the asymptotic behavior of the original function at the point $t \to \infty$ can be derived by the series $f(t) = e^{\alpha_0 t} \sum\limits_{\nu=0}^{\infty} \frac{c_\nu}{\Gamma(-\lambda_\nu)} t^{-\lambda_\nu - 1}$.

For some functions the Laplace transform inversion problem can be reduced to the problem of solving the Volterra integral equation of the first (when $x(s) = f(s)/k(s)$) or second (when $x(s) = f(s)/(1 + k(s))$) kind [12]. These equations are usually solved numerically. The inversion of the Laplace transform in UMD-spaces for resolvent families associated to an integral Volterra equation of convolution type was analyzed in [6].

The method for the mutual inversion of the Fourier-Laplace transforms was proposed by L.I. Slepyan in [7], [8]. In some cases it allows to derive the original function without usual inversion of Fourier and Laplace transforms. In more complex cases it allows to simplify the Laplace transform, which should be inverted.

The function that is presented by $F(s) = \frac{p(s)}{q(s)}$ and satisfy some conditions can be inverted with the help of residues by the second expansion theorem [9], [4]. But the analytical finding of all poles of the transform function in many cases is impossible. If $q(s)$ has distinct zeros $\alpha_k, k = \overline{1, n}$, then Heaviside's expansion formula can be used $f(t) = \sum\limits_{k=1}^{n} \frac{p(\alpha_k)}{q'(\alpha_k)} e^{\alpha_k t}$ [10]. The inverse formula $L^{-1}[F(1/s)] = \delta(t) \int\limits_0^\infty f(u)du - \frac{1}{\sqrt{t}} \int\limits_0^\infty \sqrt{u} f(u) J_1(2\sqrt{u}t)du$ was proven under some conditions in [11].

The first expansion theorem deals with the functions that can be expanded into series $F(s) = \sum\limits_{n=0}^{\infty} \frac{a_n}{s^{n+1}}$. The original function $f(t)$ can be derived in this case as $f(t) = \sum\limits_{n=0}^{\infty} \frac{a_n}{n!} t^n$ [12], [5]. Many methods were presented in [5]. In particular, the approach dealing with the functions that can be expanded into series $F(s) = \sum\limits_{n=0}^{\infty} \frac{a_n}{s^{\lambda_n}}$ or $F(s) = \sum\limits_{n=0}^{\infty} F_n(s)$ under some conditions was proposed by G. Doetsch in [5]. But there were no examples of dealing with generalized functions.

As it is seen, the problem of analytical inversion of Laplace transform is relevant and extremely important. The method, proposed in the article, can be used for some dynamic problems of elasticity, for example it can be applied for the non-stationary statement of the elastic semi-strip as development of the methodic proposed in [13].

2. Theoretical results

The present article is dedicated to the analytical inversion of Laplace transform of the following form

$$F\left(c_0+\sum_{i=1}^{N} c_i e^{-sA_i}\right) \tag{1}$$

Here $A_i > 0, i = \overline{1,N}$, $c_i, i = \overline{1,N}, c_0 \neq 0$ are real constants or functions, which do not depend on parameter of Laplace transform s, $N \geq 1$ is natural number, F is a known function.

2.1. Case 1

The inversion of (1) depend on the correspondences between $A_i, i = \overline{1,N}$. First consider the case when $A_i = n_i A_q, i = \overline{1,N}$, $n_i, i = \overline{1,N}$ are natural numbers, for some fixed number $1 \leq q \leq N$. Then the transform (1) can be rewritten in the following form

$$F\left(c_0+\sum_{i=1}^{N} c_i e^{-sn_iA_q}\right) \tag{2}$$

Denote the function of the complex variable s e^{-sA_q} as z. Since $\Re s > 0$, then $|e^{-sA_q}| = |z| < 1$. The expression (2) can be rewritten as

$$f(z) = F\left(c_0+\sum_{k=1}^{N} c_k z^{n_k}\right) \tag{3}$$

It is supposed that the function (3) satisfies Cauchy-Riemann conditions in some domain $|z| < \vartheta < 1$.

For example, if $F\left(c_0+\sum\limits_{k=1}^{N} c_k z^{n_k}\right) = \frac{1}{c_0+\sum\limits_{k=1}^{N} c_k z^{n_k}}$, than this function has $\max\limits_{1\leq k\leq N} n_k = \eta$ singular points $z_i = \alpha_i, i = \overline{1,\eta}$. So, the points $s_i =$

$-\frac{1}{A_q}\ln\alpha_i, i=\overline{1,\eta}$ are singular points for the function (2). Since γ in the formula of the inverse Laplace transform $\frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} f(s)e^{st}dt$ is the abscissa in the semi-plane of the Laplace integral's absolute convergence [5], so $\Re s > \nu > 0$, where $\nu = \max\left\{\max\limits_{1\le i\le\nu}\Re\left\{-\frac{1}{A_q}\ln\alpha_i\right\}, 0\right\}$. Thus, when $\Re s>\nu>0$ it is fulfilled that $|e^{-sA_q}| = |z| < \vartheta < 1$, where $\vartheta = e^{-\nu A_q}$. So, the function (3) in the domain $|z|<\vartheta<1$ does not have any singular points. By the the proved in [14] lemma this function satisfies Cauchy-Riemann conditions in the domain $|z|<\vartheta<1$. Some other examples of the function (3) are given in Appendix A.

Theorem 1. *If the function (3) satisfies Cauchy-Riemann conditions in some domain* $|z|<\vartheta<1$, *then* $L^{-1}\left[F\left(c_0+\sum\limits_{i=1}^{N}c_ie^{-sn_iA_q}\right)\right] = \sum\limits_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}\delta(t-kA_q)$, *where the function* $f(z)$ *has the form (3).*

Proof. **I Proof of the correctness of the function's (3) expansion into Taylor series**

By the theorem's statement the function (3) satisfies Cauchy-Riemann conditions and, therefore, it is holomorphic and regular [15] for all $|z|<\vartheta<1$.

According to the theorems [15] the regular function (3) in the circle K : $|z|<\vartheta$ can be presented by Taylor series

$$f(z) = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}z^k \tag{4}$$

Power series inside the circle of convergence can be term-by-term integrated and differentiated any number of times, moreover the radius of convergence of the derived series is equal to the radius of convergence of the original series [16].

II Application of the inverse Laplace transform to the series (4)

Thus, the series (4) has the radius of convergence $R=\vartheta$, within which this series can be term-by-term integrated. That is the following is true:

$$L^{-1}\left[F\left(c_0+\sum_{i=1}^{N}c_ie^{-sn_iA_q}\right)\right] = L^{-1}\left[\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}z^k\right] = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}\delta(t-kA_q)$$

Let's prove that the derived series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \delta(t - kA_q) \tag{5}$$

converges in the sense that all series

$$\left(\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \delta(t - kA_q), \varphi(t) \right) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \varphi(kA_q) \tag{6}$$

absolutely converge for all functions $\varphi(t) \in S_\nu \cup K^0$, where $S_\nu \subset S$, S is the main space containing all infinitely differentiable functions which when $|t| \to \infty$ tends to zero with all their derivatives of any order faster than any power $1/|t|$ [17], S_ν contains such infinitely differentiable functions that when $t \to +\infty$ tends to zero with all their derivatives of any order faster than $e^{-\nu t}$, K^0 is the main space containing all continuous functions that are zero outside some bounded domain [17]. Obviously, if the absolute convergence of series (6) is proved for all functions from the spaces S_ν and K^0, then it will also take place for the functions from the main spaces $K^m, m > 0, K$, since $K \subset K^m \subseteq K^0$ [17].

Let's prove the convergence of the following series

$$\sum_{k=0}^{\infty} \frac{\left| f^{(k)}(0) \right|}{k!} \left| \varphi(kA_q) \right| \tag{7}$$

III Proof of the series' (7) convergence for $\varphi(t) \in S_\nu$

According to [18] if the limit $\lim\limits_{n \to \infty} \frac{a_n}{b_n} = K < \infty$ exists then the convergence of the series $\sum\limits_{n=1}^{\infty} b_n$ with positive terms implies the convergence of the series $\sum\limits_{n=1}^{\infty} a_n$ with positive terms.

Let's make a comparison with the series

$$\sum_{k=0}^{\infty} \frac{\left| f^{(k)}(0) \right|}{k!} z_0^k, \tag{8}$$

which is the series with positive terms. By Abel's theorem [16], the convergence of the series (4) in the circle $K : |z| < \vartheta$ implies the convergence of the series (8) when $0 < z_0 < \vartheta$. Let's set $z_0 = e^{-\nu A_q} - \varepsilon_0$ for some small fixed $\varepsilon_0 > 0$. Since $\vartheta = e^{-\nu A_q}$ and $\varepsilon_0 > 0$ is small, then $0 < z_0 < \vartheta$.

Let's prove that

$$\lim_{k\to\infty} \frac{\frac{\left|f^{(k)}(0)\right|}{k!}\left|\varphi(kA_q)\right|}{\frac{\left|f^{(k)}(0)\right|}{k!} z_0^k} = \lim_{k\to\infty} \frac{\left|\varphi(kA_q)\right|}{z_0^k} < \infty \tag{9}$$

for the functions $\varphi(t) \in S_\nu$.

Let's rewrite the limit (9) in the following form $\lim\limits_{k\to\infty} \frac{|\varphi(kA_q)|}{\left(e^{-\nu A_q}-\varepsilon_0\right)^k}$ or, the same, $\lim\limits_{k\to\infty} \frac{|\varphi(kA_q)|}{e^{-k\nu A_q}\left(1-\frac{\varepsilon_0}{e^{-\nu A_q}}\right)^k}$.

Accordingly to [19] $(1+x)^n \geq 1+nx, x > -1, n > 1$. Note that this inequality also holds when $n=0$ and $n=1$. Thus,

$$0 \leq \frac{|\varphi(kA_q)|}{e^{-k\nu A_q}\left(1-\frac{\varepsilon_0}{e^{-\nu A_q}}\right)^k} \leq \frac{|\varphi(kA_q)|}{e^{-k\nu A_q}\left(1-k\frac{\varepsilon_0}{e^{-\nu A_q}}\right)} \tag{10}$$

since from $z_0 = e^{-\nu A_q} - \varepsilon_0 > 0$ it follows that $\frac{\varepsilon_0}{e^{-\nu A_q}} < 1$ and $-\frac{\varepsilon_0}{e^{-\nu A_q}} > -1$.

Due to the fact that $\varphi(t) \in S_\nu$, $\varphi(kA_q)$ decreases on $+\infty$ faster than $e^{-\nu k A_q}$. So, $\lim\limits_{k\to\infty} \frac{|\varphi(kA_q)|}{e^{-k\nu A_q}} = 0$. And $\lim\limits_{k\to\infty}\left(1-k\frac{\varepsilon_0}{e^{-\nu A_q}}\right) = \infty$. Then by the theorem of the limit of the quotient [20] it is derived that

$$\lim_{k\to\infty} \frac{|\varphi(kA_q)|}{e^{-k\nu A_q}\left(1-k\frac{\varepsilon_0}{e^{-\nu A_q}}\right)} = 0 \tag{11}$$

Thus from (10) with regard to (11) by the property of comparison of limits [20] it is derived that $\lim\limits_{k\to\infty} \frac{|\varphi(kA_q)|}{e^{-k\nu A_q}\left(1-\frac{\varepsilon_0}{e^{-\nu A_q}}\right)^k} = 0 < \infty$. That is (9) holds. Then by the theorem the series (7) converges for all functions $\varphi(t) \in S_\nu$.

IV Proof of the series' (7) convergence for $\varphi(t) \in K^0$

Note that for the functions $\varphi(t) \in K^0$, since they are equal to zero outside some bounded domain, there exists a number N such that $|\varphi(kA_q)| = 0$ for $k > N$. In this case, the convergence of the series (7) can be proved by another theorem, according to which if, at least starting from some place (say, for $n > N$), the inequality $a_n \leq b_n$ holds, then the convergence of the series $\sum\limits_{n=1}^{\infty} b_n$ with positive terms implies the convergence of the series $\sum\limits_{n=1}^{\infty} a_n$ with positive terms [18]. Then for $k > N$ the following correspondence takes place $0 = \frac{|f^{(k)}(0)|}{k!}|\varphi(kA_q)| \leq \frac{|f^{(k)}(0)|}{k!} z_0^k$. Hence the series (7) is convergent for all functions $\varphi(t) \in K^0$. Thus, it is proved that the series (6) converges absolutely for all functions $\varphi(t) \in S_\nu \cup K^0$, and the series (5) converges in the sense indicated earlier.

The proved convergence of the series (5) implies the correctness of the term-by-term application of the series (5) to any function from the spaces $K^m, m \geq 0, K, S_\nu$.

V Proof that the resulting series (5) is the original for the Laplace transform (2)

Now let's prove that the resulting series (5) is the original for the Laplace transform (2). For this, the Laplace transform is applied to the series (5)

$$L\left[\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\delta(t-kA_q)\right] = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}e^{-skA_q}$$

Let's prove that the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}e^{-skA_q} \tag{12}$$

converges to the known transform (2).

The series (12), taking into account the change of variables $z = e^{-sA_q}$, can be written as (4), that is, it is an expansion of the function $f(z)$ (3) in Taylor series. According to the theorems [15] and the proved regularity of the function $f(z)$, it is derived that the series (12) converges to the function $f(z)$ (3) with the radius of convergence $R = \vartheta$, which corresponds to the entire range of the variable $|z| < \vartheta$.

The statement of the theorem is proved.

2.2. Case 2

Let's consider the most general case when $A_i = \sum\limits_{j=1}^{m} n_{ij}A_{q_j}, i = \overline{1,N}, n_{ij}, i = \overline{1,N}, j = \overline{1,m}, m > 1$ are natural numbers, for some fixed numbers $1 \leq q_j \leq N$, moreover $A_{q_j} \neq A_{q_k}, j \neq k, j,k = \overline{1,m}$. Then the transform (1) can be rewritten as

$$F\left(c_0 + \sum_{i=1}^{N} c_i e^{-s\sum\limits_{j=1}^{m} n_{ij}A_{q_j}}\right) \tag{13}$$

Denote the functions of the complex variable s as $z_j = e^{-sA_{q_j}}, j = \overline{1,m}$. Since $\Re s > 0$, then $|e^{-sA_{q_j}}| = |z_j| < 1$. The expression (13) can be rewritten as

$$f(z_1, \dots, z_m) = F\left(c_0 + \sum_{k=1}^{N} c_k \prod_{j=1}^{m} z_j^{n_{kj}}\right) \tag{14}$$

It is supposed that the function (14) satisfies Cauchy-Riemann conditions in some domain $|z_j| < \vartheta_j < 1, j = \overline{1, m}$.

For example, if $F\left(c_0 + \sum\limits_{k=1}^{N} c_k \prod\limits_{j=1}^{m} z_j^{n_{kj}}\right) = \frac{1}{c_0 + \sum\limits_{k=1}^{N} c_k \prod\limits_{j=1}^{m} z_j^{n_{kj}}}$, than this function has $\max\limits_{1 \le k \le N} n_{k1} = \eta$ singular points $z_{i1} = \alpha_i(z_2, \dots, z_m), i = \overline{1, \eta}$. So, the points $s_i = \nu_i, i = \overline{1, \eta}$ that can be found from the equation $s_i = -\frac{1}{A_{q_1}} \ln \alpha_i \left(e^{-s_i A_{q_2}}, \dots, e^{-s_i A_{q_m}}\right), i = \overline{1, \eta}$ are singular points for the function (13). Since γ in the formula of the inverse Laplace transform $\frac{1}{2\pi i} \int\limits_{\gamma - i\infty}^{\gamma + i\infty} f(s) e^{st} ds$ is the abscissa in the semi-plane of the Laplace integral's absolute convergence [5], so $\Re s > \nu > 0$, where $\nu = \max\{\max\limits_{1 \le i \le \eta} \nu_i, 0\}$. Thus, when $\Re s > \nu > 0$ it is fulfilled that $|e^{-sA_{q_j}}| = |z_j| < \vartheta_j < 1, j = \overline{1, m}$, where $\vartheta_j = e^{-\nu A_{q_j}}, j = \overline{1, m}$. So, the function (14) in the domain $|z_j| < \vartheta_j < 1, j = \overline{1, m}$ does not have any singular points.

Theorem 2. *If the function (3) satisfies Cauchy-Riemann conditions in some domain* $|z_j| < \vartheta_j < 1, j = \overline{1, m}$, *then* $L^{-1}\left[F\left(c_0 + \sum\limits_{i=1}^{N} c_i e^{-s \sum\limits_{j=1}^{m} n_{ij} A_{q_j}}\right)\right] = \sum\limits_{k_1, \dots, k_m = 0}^{\infty} \frac{1}{k_1! \dots k_m!} \frac{\partial^{k_1 + \dots + k_m} f(0, \dots, 0)}{\partial z_1^{k_1} \dots \partial z_m^{k_m}} \delta(t - k_1 A_{q_1} - \dots - k_m A_{q_m})$, *where* $f(z_1, \dots, z_m)$ *has the form (14).*

Proof. **I Proof of the correctness of the function's (14) expansion into Taylor series**

First let's prove that the function (14) is holomorphic. By the Hartogs-Osgood theorem [21] a complex-valued function $f(x_1, ..., x_m)$ is holomorphic on an open set $U \subset \mathbb{C}^m$ (here $\mathbb{C}$ is the complex space) if, for each point $a = (a_1, ..., a_m) \in U$ and each number $j (1 \le j \le m)$, the function $f(a_1, ..., a_{j-1}, x_j, a_{j+1}, ..., a_m)$ of one complex variable x_j defined on the open set $\{x_j \in \mathbb{C} | (a_1, ..., a_{j-1}, x_j, a_{j+1}, ..., a_m) \in U\} \subset \mathbb{C}^m$, is holomorphic on the indicated open sets of the space $\mathbb{C}$.

Let's consider m functions $f(a_1,...,a_{j-1},z_j,a_{j+1},...,a_m), j=\overline{1,m}$, where $a_j\in\mathbb{C}, j=\overline{1,m}$ are arbitrary points for which it holds that $|a_j|<\vartheta_j<1, j=\overline{1,m}$, and prove that they all satisfy Cauchy-Riemann conditions.

$$f(a_1,...,a_{j-1},z_j,a_{j+1},...,a_m)=F\left(d_0+\sum_{k=1}^{N}d_k z_j^{n_{k_j}}\right), j=\overline{1,m}$$

where $d_0=c_0, d_k=c_k\prod\limits_{j=1,i\neq j}^{m}a_i^{n_{ki}}, k=\overline{1,N}$.

Note that this function coincides with the function $f(z_j)=F\left(c_0+\sum\limits_{k=1}^{N}c_k z_j^{n_k}\right)$ (3) which by the theorem's condition satisfies Cauchy-Riemann conditions in the domain $|z_j|<\vartheta_j<1, j=\overline{1,m}$. So, according to [22] all functions $f(a_1,...,a_{j-1},z_j,a_{j+1},...,a_m), j=\overline{1,m}$ are holomorphic when $|z_j|<\vartheta_j<1, j=\overline{1,m}$ for any points $a_j\in\mathbb{C}, j=\overline{1,m}$ such that $|a_j|<\vartheta_j<1, j=\overline{1,m}$. Hence, by the Hartogs-Osgood theorem, the function (14) is holomorphic on the open set $P=\left\{(z_1,...,z_m)\in C^m||z_j|<\vartheta_j, j=\overline{1,m}\right\}$.

According to the theorem [21] the holomorphic in an open polycylinder $P=\left\{(z_1,...,z_m)\in\mathbb{C}^m||z_j|<\vartheta_j, j=\overline{1,m}\right\}$ function (14) is uniquely expanded into the absolutely convergent Taylor series

$$\sum_{k_1,...,k_m=0}^{\infty}\frac{1}{k_1!...k_m!}\frac{\partial^{k_1+...+k_m}f(0,...,0)}{\partial z_1^{k_1}...\partial z_m^{k_m}}\prod_{j=1}^{m}z_j^{k_j} \tag{15}$$

II Application of the inverse Laplace transform to the series (15)

Accordingly, the following is true:

$$L^{-1}\left[F\left(c_0+\sum_{i=1}^{N}c_i e^{-s\sum\limits_{j=1}^{m}n_{ij}A_{q_j}}\right)\right]=$$
$$=L^{-1}\left[\sum_{k_1,...,k_m=0}^{\infty}\frac{1}{k_1!...k_m!}\frac{\partial^{k_1+...+k_m}f(0,...,0)}{\partial z_1^{k_1}...\partial z_m^{k_m}}\prod_{j=1}^{m}z_j^{k_j}\right]=$$
$$=\sum_{k_1,...,k_m=0}^{\infty}\frac{1}{k_1!...k_m!}\frac{\partial^{k_1+...+k_m}f(0,...,0)}{\partial z_1^{k_1}...\partial z_m^{k_m}}\delta(t-k_1A_{q_1}-...-k_mA_{q_m})$$

Let's prove that the derived series

$$\sum_{k_1,...,k_m=0}^{\infty}\frac{1}{k_1!...k_m!}\frac{\partial^{k_1+...+k_m}f(0,...,0)}{\partial z_1^{k_1}...\partial z_m^{k_m}}\delta(t-k_1A_{q_1}-...-k_mA_{q_m}) \tag{16}$$

converges in the sense that all series

$$\left(\sum_{k_1,...,k_m=0}^{\infty} \frac{1}{k_1!...k_m!} \frac{\partial^{k_1+...+k_m} f(0,...,0)}{\partial z_1^{k_1}...\partial z_m^{k_m}} \delta(t - k_1 A_{q_1} - ... - k_m A_{q_m}), \varphi(t) \right) =$$
$$= \sum_{k_1,...,k_m=0}^{\infty} \frac{1}{k_1!...k_m!} \frac{\partial^{k_1+...+k_m} f(0,...,0)}{\partial z_1^{k_1}...\partial z_m^{k_m}} \varphi(k_1 A_{q_1} + ... + k_m A_{q_m}) \tag{17}$$

absolutely converge for all functions $\varphi(t) \in S_\nu \cup K^0$, where S_ν and K^0 are the spaces described in the theorem 1.

Let's prove the convergence of the following series

$$\sum_{k_1,...,k_m=0}^{\infty} \frac{1}{k_1!...k_m!} \left| \frac{\partial^{k_1+...+k_m} f(0,...,0)}{\partial z_1^{k_1}...\partial z_m^{k_m}} \right| |\varphi(k_1 A_{q_1} + ... + k_m A_{q_m})| \tag{18}$$

III Proof of the series' (18) convergence for $\varphi(t) \in S_\nu$

III.1 Proof of the limit case theorem for multiple series' convergence

According to [18] and the theorem [23] if for two multiple series $\sum\limits_{k_1,...,k_m=0}^{\infty} u_{k_1,..,k_m}$ and $\sum\limits_{k_1,...,k_m=0}^{\infty} v_{k_1,..,k_m}$ with positive terms there are such $k_{01},..,k_{0m}$ that when $k_i > k_{0i}, i = \overline{1,m}$ the inequalities $u_{k_1,..,k_m} \leq v_{k_1,..,k_m}$ hold, then the convergence of the multiple series $\sum\limits_{k_1,...,k_m=0}^{\infty} v_{k_1,..,k_m}$ implies the convergence of the multiple series $\sum\limits_{k_1,...,k_m=0}^{\infty} u_{k_1,..,k_m}$. Also the limit case of this theorem can be formulated. If the multiple limit $\lim\limits_{k_1,...,k_m \to \infty} \frac{u_{k_1,..,k_m}}{v_{k_1,..,k_m}} = K < \infty$, then the convergence of the multiple series $\sum\limits_{k_1,...,k_m=0}^{\infty} v_{k_1,..,k_m}$ implies the convergence of the multiple series $\sum\limits_{k_1,...,k_m=0}^{\infty} u_{k_1,..,k_m}$. Indeed, if $\lim\limits_{k_1,...,k_m \to \infty} \frac{u_{k_1,..,k_m}}{v_{k_1,..,k_m}} = K < \infty$ then by the definition of the multiple limit [20] the following holds: for each $\varepsilon > 0$, no matter how small it may be, there exists a number N such that for all $k_i > N, i = \overline{1,m}$: $|\frac{u_{k_1,..,k_m}}{v_{k_1,..,k_m}} - K| < \varepsilon$ or $\frac{u_{k_1,..,k_m}}{v_{k_1,..,k_m}} < K + \varepsilon$. That is the following estimation holds $u_{k_1,..,k_m} < (K+\varepsilon) v_{k_1,..,k_m}$. By the theorem of the multiplication of the multiple series by the digit [23], the series $\sum\limits_{k_1,...,k_m=0}^{\infty} (K+\varepsilon) v_{k_1,..,k_m}$ converges. Then by the theorem indicated earlier the series $\sum\limits_{k_1,...,k_m=0}^{\infty} u_{k_1,..,k_m}$ converges.

III.2 Comparison with the convergent series

Let's make a comparison with the series

$$\sum_{k_1,...,k_m=0}^{\infty} \frac{1}{k_1!...k_m!} \left| \frac{\partial^{k_1+...+k_m} f(0,...,0)}{\partial z_1^{k_1}...\partial z_m^{k_m}} \right| \prod_{j=1}^{m} z_{0j}^{k_j}, \tag{19}$$

which is the series with positive terms. The absolute convergence of the series (15) in the polycylinder $P = \left\{(z_1,...,z_m) \in \mathbb{C}^m \| z_j| < \vartheta_j, j = \overline{1,m}\right\}$ implies the absolute convergence of the series (19) when $0 < z_{0j} < \vartheta_j, j = \overline{1,m}$. Let's set $z_{0j} = e^{-\nu A_{q_j}} - \varepsilon_j, j = \overline{1,m}$ for some small fixed $\varepsilon_j > 0, j = \overline{1,m}$. Since $\vartheta_j = e^{-\nu A_{q_j}}, j = \overline{1,m}$ and $\varepsilon_j > 0, j = \overline{1,m}$ are small, then $0 < z_{0j} < \vartheta_j, j = \overline{1,m}$.

Let's prove that

$$\lim_{k_1,...,k_m\to\infty} \frac{\frac{1}{k_1!...k_m!}\left|\frac{\partial^{k_1+...+k_m} f(0,...,0)}{\partial z_1^{k_1}...\partial z_m^{k_m}}\right| \left|\varphi(k_1A_{q_1}+...+k_mA_{q_m})\right|}{\frac{1}{k_1!...k_m!}\left|\frac{\partial^{k_1+...+k_m} f(0,...,0)}{\partial z_1^{k_1}...\partial z_m^{k_m}}\right| \prod_{j=1}^{m} z_{0j}^{k_j}} = \tag{20}$$
$$= \lim_{k_1,...,k_m\to\infty} \frac{\left|\varphi(k_1A_{q_1}+...+k_mA_{q_m})\right|}{\prod_{j=1}^{m} z_{0j}^{k_j}} < \infty$$

for the functions $\varphi(t) \in S_\nu$.

Let's rewrite the limit (20) in the following form $\lim\limits_{k_1,...,k_m\to\infty} \frac{\left|\varphi(k_1A_{q_1}+...+k_mA_{q_m})\right|}{\prod_{j=1}^{m}\left(e^{-\nu A_{q_j}}-\varepsilon_j\right)^{k_j}}$ or, the same, $\lim\limits_{k_1,...,k_m\to\infty} \frac{\left|\varphi(k_1A_{q_1}+...+k_mA_{q_m})\right|}{\prod_{j=1}^{m} e^{-k_j\nu A_{q_j}} \prod_{j=1}^{m}\left(1-\frac{\varepsilon_j}{e^{-\nu A_{q_j}}}\right)^{k_j}}$.

Accordingly to [19] $(1+x)^n \geq 1+nx, x > -1, n > 1$. Note that this inequality also holds when $n = 0$ and $n = 1$. Thus,

$$0 \leq \frac{|\varphi(k_1A_{q_1}+...+k_mA_{q_m})|}{\prod_{j=1}^{m} e^{-k_j\nu A_{q_j}} \prod_{j=1}^{m}\left(1-\frac{\varepsilon_j}{e^{-\nu A_{q_j}}}\right)^{k_j}} \leq \frac{|\varphi(k_1A_{q_1}+...+k_mA_{q_m})|}{\prod_{j=1}^{m} e^{-k_j\nu A_{q_j}} \prod_{j=1}^{m}\left(1-k_j\frac{\varepsilon_j}{e^{-\nu A_{q_j}}}\right)} \tag{21}$$

since from $z_{0j} = e^{-\nu A_{q_j}} - \varepsilon_j > 0, j = \overline{1,m}$ it follows that $\frac{\varepsilon_j}{e^{-\nu A_{q_j}}} < 1, j = \overline{1,m}$ and $-\frac{\varepsilon_j}{e^{-\nu A_{q_j}}} > -1, j = \overline{1,m}$.

Due to the fact that $\varphi(t) \in S_\nu$, $\varphi(k_1A_{q_1}+...+k_mA_{q_m})$ decreases on $+\infty$ faster than $e^{-\nu(k_1A_{q_1}+...+k_mA_{q_m})}$. So, $\lim\limits_{k_1,...,k_m\to\infty} \frac{\left|\varphi(k_1A_{q_1}+...+k_mA_{q_m})\right|}{\prod_{j=1}^{m} e^{-k_j\nu A_{q_j}}} = 0$. And by the theorem of limit of the product [20] $\lim\limits_{k_1,...,k_m\to\infty} \prod_{j=1}^{m}\left(1-k_j\frac{\varepsilon_j}{e^{-\nu A_{q_j}}}\right) = \infty$.

Then by the theorem of the limit of the quotient [20] it is derived that

$$\lim_{k_1,\dots,k_m\to\infty} \frac{|\varphi(k_1 A_{q_1} + \dots + k_m A_{q_m})|}{\prod\limits_{j=1}^{m} e^{-k_j \nu A_{q_j}} \prod\limits_{j=1}^{m} \left(1 - k_j \frac{\varepsilon_j}{e^{-\nu A_{q_j}}}\right)} = 0 \tag{22}$$

III.3 Proof of the comparison theorem for multiple limits

Let's prove for the multiple limits the following comparison theorem. If for the sequences $x_{k_1,\dots,k_m}, y_{k_1,\dots,k_m}, z_{k_1,\dots,k_m}$ the inequalities $x_{k_1,\dots,k_m} \le y_{k_1,\dots,k_m} \le z_{k_1,\dots,k_m}$ always hold, and the sequences $x_{k_1,\dots,k_m}, z_{k_1,\dots,k_m}$ tend to the common multiple limit $\lim\limits_{k_1,\dots,k_m\to\infty} x_{k_1,\dots,k_m} = \lim\limits_{k_1,\dots,k_m\to\infty} z_{k_1,\dots,k_m} = a$, then the sequence $y_{k_1,\dots,k_m}$ also has the same multiple limit $\lim\limits_{k_1,\dots,k_m\to\infty} y_{k_1,\dots,k_m} = a$. Let's fix some arbitrary $\varepsilon > 0$. For it there is some number N_1 that when $k_i > N_1, i = \overline{1,m}$ the following holds $a-\varepsilon < x_{k_1,\dots,k_m} < a+\varepsilon$. Also there is some number N_2 that when $k_i > N_1, i = \overline{1,m}$ the following holds $a - \varepsilon < z_{k_1,\dots,k_m} < a + \varepsilon$. Choosing $N > \max\{N_1, N_2\}$ for $k_i > N, i = \overline{1,m}$ both previous double inequalities hold and then $a - \varepsilon < x_{k_1,\dots,k_m} \le y_{k_1,\dots,k_m} \le z_{k_1,\dots,k_m} < a + \varepsilon$. Thus,

$$a - \varepsilon < y_{k_1,\dots,k_m} < a + \varepsilon \quad \text{or} \quad |y_{k_1,\dots,k_m} - a| < \varepsilon$$

when $k_i > N$, $i = \overline{1,m}$. That is

$$\lim_{k_1,\dots,k_m\to\infty} y_{k_1,\dots,k_m} = a$$

is proved.

Thus from (21) with regard to (22) by the proven property of comparison of multiple limits it is derived that

$$\lim_{k_1,\dots,k_m\to\infty} \frac{|\varphi(k_1 A_{q_1} + \dots + k_m A_{q_m})|}{\prod\limits_{j=1}^{m} e^{-k_j \nu A_{q_j}} \prod\limits_{j=1}^{m} \left(1 - \frac{\varepsilon_j}{e^{-\nu A_{q_j}}}\right)^{k_j}} = 0 < \infty.$$

That is (20) holds. Then by the theorem the series (18) converges for all functions $\varphi(t) \in S_\nu$.

IV Proof of the series' (18) convergence for $\varphi(t) \in K^0$

Note that for the functions $\varphi(t) \in K^0$, since they are equal to zero outside some bounded domain, there exist numbers $k_{01}, \dots, k_{0m}$ such that

$$|\varphi(k_1 A_{q_1} + \dots + k_m A_{q_m})| = 0$$

for $k_i > k_{0i}$, $i = \overline{1,m}$. In this case the convergence of the series (18) can be proved by the indicated earlier theorem of the comparison of the multiple series with positive terms. Then for $k_i > k_{0i}, i = \overline{1,m}$:

$$0 = \frac{1}{k_1!...k_m!}\left|\frac{\partial^{k_1+...+k_m} f(0,...,0)}{\partial z_1^{k_1}...\partial z_m^{k_m}}\right| |\varphi(k_1 A_{q_1} + ... + k_m A_{q_m})| \leq$$

$$\leq \frac{1}{k_1!...k_m!}\left|\frac{\partial^{k_1+...+k_m} f(0,...,0)}{\partial z_1^{k_1}...\partial z_m^{k_m}}\right| \prod_{i=1}^{m} z_{0i}^{k_i}$$

is derived. Therefore, the series (18) is convergent for all functions $\varphi(t) \in K^0$. Thus, it is proved that the series (17) converges absolutely for all functions $\varphi(t) \in S_\nu \cup K^0$, and the series (16) converges in the sense indicated earlier.

The proved convergence of the series (16) implies the correctness of the term-by-term application of the series (16) to any function from the spaces $K^m, m \geq 0, K, S_\nu$.

V Proof that the resulting series (16) is the original for the Laplace transform (13)

Now let's prove that the resulting series (16) is the original for the Laplace transform (13). For this, the Laplace transform is applied to the series (16)

$$L\left[\sum_{k_1,...,k_m=0}^{\infty} \frac{1}{k_1!...k_m!} \frac{\partial^{k_1+...+k_m} f(0,...,0)}{\partial z_1^{k_1}...\partial z_m^{k_m}} \delta(t - k_1 A_{q_1} - ... - k_m A_{q_m})\right] =$$

$$= \sum_{k_1,...,k_m=0}^{\infty} \frac{1}{k_1!...k_m!} \frac{\partial^{k_1+...+k_m} f(0,...,0)}{\partial z_1^{k_1}...\partial z_m^{k_m}} e^{-s\sum\limits_{j=1}^{m} k_j A_{q_j}}$$

Let's prove that the derived series

$$\sum_{k_1,...,k_m=0}^{\infty} \frac{1}{k_1!...k_m!} \frac{\partial^{k_1+...+k_m} f(0,...,0)}{\partial z_1^{k_1}...\partial z_m^{k_m}} e^{-s\sum\limits_{j=1}^{m} k_j A_{q_j}} \tag{23}$$

converges to the known transform (13).

The series (23), taking into account the change of variables $z_j = e^{-sA_{q_j}}, j = \overline{1,m}$, can be written as (15), that is, it is an expansion of the function $f(z_1,...,z_m)$ (14) in Taylor series. According to the theorems [21] and the proved holomorphy of the function $f(z_1,...,z_m)$, it is derived that the series (23) converges to the function $f(z_1,...,z_m)$ (14) with the radiuses of convergence $r_j = \vartheta_j, j = \overline{1,m}$, which corresponds to the entire range of the variables $|z_j| < \vartheta_j, j = \overline{1,m}$.

The statement of the theorem is proved.

2.3. Relation with the convolution

Let's consider the function of the structure (1)

$$\frac{1}{c_0 + \sum\limits_{i=1}^{N} c_i e^{-sA_i}} \tag{24}$$

and the most general form of the transform, the partial case of which is the function (24)

$$x^L(s) = \frac{f^L(s)}{c_0 + K^L(s)} \tag{25}$$

Here $f(t) = \delta(t), K(t) = \sum\limits_{i=1}^{N} c_i \delta(t - A_i)$ for the function (24). As it was shown in [14], the equation (25) can be written using convolution [17]

$$\left[c_0 \delta(t) + \sum_{i=1}^{N} c_i \delta(t - A_i) \right] * x(t) = \delta(t) \tag{26}$$

That is, finding the original $x(t)$ is reduced to the solving of the convolution equation (26). So, the derived results from the theorems regarding the function (24) can be verified using the convolution. Also the following consequences can be formulated

Consequence 1 $\left[c_0 \delta(t) + \sum\limits_{i=1}^{N} c_i \delta(t - n_i A_m) \right]^{-1} = \sum\limits_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \delta(t - kA_m)$, *where* $f(z) = \frac{1}{c_0 + \sum\limits_{k=1}^{N} c_k z^{n_k}}$.

Consequence 2

$$\left[c_0 \delta(t) + \sum_{i=1}^{N} c_i \delta(t - k_1 A_{q_1} - \ldots - k_m A_{q_m}) \right]^{-1}$$
$$= \sum_{k_1, \ldots, k_m = 0}^{\infty} \frac{1}{k_1! \ldots k_m!} \frac{\partial^{k_1 + \ldots + k_m} f(0, \ldots, 0)}{\partial z_1^{k_1} \ldots \partial z_m^{k_m}} \delta(t - k_1 A_{q_1} - \ldots - k_m A_{q_m}),$$

where $f(z_1, \ldots, z_m) = \frac{1}{c_0 + \sum\limits_{k=1}^{N} c_k \prod\limits_{j=1}^{m} z_j^{n_{kj}}}$.

The verification of the theorems for some examples of the function (24) using given consequences is done in Appendix B.

3. Conclusions

In the article the new method for the analytical inversion of the Laplace transform is proposed for some cases. The theorems are proved. The results derived by the new method are compared with the formulas known in literature. The new formulas of analytical inversion of Laplace transform are presented. This method can be used for the mechanical problems dealing with Laplace transform.

A. Some other examples of functions of the structure (2)

A.1. Logarithmic case

The transform (2) can be written in the following form

$$\ln\left|c_0+\sum_{i=1}^{N} c_i e^{-sn_iA_q}\right| \tag{A.1}$$

The function (3) in this case can be written as

$$f(z)=\ln\left|c_0+\sum_{k=1}^{N} c_k z^{n_k}\right| \tag{A.2}$$

This function has $\max\limits_{1\le k\le N} n_k = \eta$ singular points $z_i = \alpha_i, i = \overline{1,\eta}$. So, the points $s_i = -\frac{1}{A_q}\ln\alpha_i, i = \overline{1,\eta}$ are singular points for the function (A.1). Since γ in the formula of the inverse Laplace transform $\frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} f(s)e^{st}dt$ is the abscissa in the semi-plane of the Laplace integral's absolute convergence [5], so $\Re s > \nu > 0$, where $\nu = \max\left\{\max\limits_{1\le i\le \nu}\Re\left\{-\frac{1}{A_q}\ln\alpha_i\right\},0\right\}$. Thus, when $\Re s > \nu > 0$ it is fulfilled that $|e^{-sA_q}| = |z| < \vartheta < 1$, where $\vartheta = e^{-\nu A_q}$. So, the function (A.2) in the domain $|z| < \vartheta < 1$ does not have any singular points.

Lemma 1 The function (A.2) satisfies Cauchy-Riemann conditions in the domain $|z| < \vartheta < 1$ where it has no singular points.

Proof. Cauchy-Riemann conditions for the function $f(z) = u(x,y) + iv(x,y)$ have the following form [15]:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y};\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \tag{A.3}$$

The function (A.2) can be rewritten in the following form

$$f(z) = \ln\left|c_0 + \sum_{k=1}^{N} c_k z^{n_k}\right| = \begin{cases} \ln\left(c_0 + \sum\limits_{k=1}^{N} c_k (x+iy)^{n_k}\right), c_0 + \sum\limits_{k=1}^{N} c_k (x+iy)^{n_k} > 0, \\ \ln\left(-c_0 - \sum\limits_{k=1}^{N} c_k (x+iy)^{n_k}\right), c_0 + \sum\limits_{k=1}^{N} c_k (x+iy)^{n_k} < 0 \end{cases} \tag{A.4}$$

Calculate partial derivatives of the first function in (A.4):

$$\frac{\partial f}{\partial x} = \frac{\sum\limits_{k=1}^{N} c_k n_k (x+iy)^{n_k-1}}{c_0 + \sum\limits_{k=1}^{N} c_k (x+iy)^{n_k}}; \frac{\partial f}{\partial y} = \frac{\sum\limits_{k=1}^{N} c_k n_k i (x+iy)^{n_k-1}}{c_0 + \sum\limits_{k=1}^{N} c_k (x+iy)^{n_k}} \tag{A.5}$$

Note that partial derivatives of the second function in (A.4) have the same form (A.5).

Let's rewrite the denominator

$$\frac{1}{c_0+\sum\limits_{k=1}^{N} c_k z^{n_k}} = \frac{1}{c_0+\sum\limits_{k=1}^{N} c_k (x+iy)^{n_k}} = \frac{1}{c_0+\sum\limits_{k=1}^{N} c_k \sum\limits_{l=0}^{n_k} C_{n_k}^{l} x^{n_k-l}(iy)^l} =$$
$$= \frac{1}{c_0+\sum\limits_{k=1}^{N} c_k \sum\limits_{l=0}^{[n_k/2]} C_{n_k}^{2l} x^{n_k-2l}(-1)^l y^{2l} + i\sum\limits_{k=1}^{N} c_k \sum\limits_{l=0}^{[(n_k-1)/2]} C_{n_k}^{2l+1} x^{n_k-2l-1}(-1)^l y^{2l+1}} =$$
$$= \frac{1}{Re+iIm} = \frac{Re-iIm}{Re^2+Im^2}$$

Here

$$\begin{aligned} Re(x,y) &= c_0 + \sum_{k=1}^{N} c_k \sum_{l=0}^{[n_k/2]} C_{n_k}^{2l} x^{n_k-2l}(-1)^l y^{2l}, \\ Im(x,y) &= \sum_{k=1}^{N} c_k \sum_{l=0}^{[(n_k-1)/2]} C_{n_k}^{2l+1} x^{n_k-2l-1}(-1)^l y^{2l+1}, \end{aligned} \tag{A.6}$$

where $[n_k/2]$ and $[(n_k-1)/2]$ are integer parts of division.

Analogically to the denominator, the nominator can be rewritten as

$$\sum_{k=1}^{N} c_k n_k (x+iy)^{n_k-1} = \sum_{k=1}^{N} c_k n_k \sum_{l=0}^{n_k-1} C_{n_k-1}^{l} x^{n_k-l-1}(iy)^l =$$
$$= \sum_{k=1}^{N} c_k n_k \sum_{l=0}^{[(n_k-1)/2]} C_{n_k-1}^{2l} x^{n_k-2l-1}(-1)^l y^{2l}$$

$$
\begin{aligned}
&+ i \sum_{k=1}^{N} c_k n_k \sum_{l=0}^{[(n_k-2)/2]} C_{n_k-1}^{2l+1} x^{n_k-2l-2}(-1)^l y^{2l+1} \\
&= re + i\, im,
\end{aligned}
$$

where

$$
\begin{aligned}
re(x,y) &= \sum_{k=1}^{N} c_k n_k \sum_{l=0}^{[(n_k-1)/2]} C_{n_k-1}^{2l} x^{n_k-2l-1}(-1)^l y^{2l}, \\
im(x,y) &= \sum_{k=1}^{N} c_k n_k \sum_{l=0}^{[(n_k-2)/2]} C_{n_k-1}^{2l+1} x^{n_k-2l-2}(-1)^l y^{2l+1}
\end{aligned}
$$

Then the partial derivatives (A.5) can be rewritten in the following form

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{(re + iim)(Re - i\, Im)}{Re^2 + Im^2} = \\
&= \frac{re\, Re + im\, Im}{Re^2 + Im^2} + i \frac{im\, Re - re\, Im}{Re^2 + Im^2} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \\
\frac{\partial f}{\partial y} &= \frac{(i\, re - im)(Re - i\, Im)}{Re^2 + Im^2} = \\
&= \frac{-im\, Re + re\, Im}{Re^2 + Im^2} + i \frac{re\, Re + im\, Im}{Re^2 + Im^2} = \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}.
\end{aligned}
$$

Here

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{reRe + imIm}{Re^2 + Im^2}, & \frac{\partial v}{\partial x} &= \frac{imRe - reIm}{Re^2 + Im^2}, \\
\frac{\partial u}{\partial y} &= \frac{-imRe + reIm}{Re^2 + Im^2}, & \frac{\partial v}{\partial y} &= \frac{reRe + imIm}{Re^2 + Im^2},
\end{aligned}
$$

so it is seen that Cauchy-Riemann conditions (A.3) are fulfilled for both functions in (A.4). Consequently, it is derived that the function (A.2) satisfies Cauchy-Riemann conditions (A.3) for all $|z| < \vartheta < 1$.

A.2. Trigonometric case

The transform (2) can be written in the following form

$$
\sin\left(c_0 + \sum_{i=1}^{N} c_i e^{-sn_iA_q}\right) \tag{A.7}
$$

or

$$
\cos\left(c_0 + \sum_{i=1}^{N} c_i e^{-sn_iA_q}\right) \tag{A.8}
$$

The function (3) in this case can be written as

$$f(z) = \sin\left(c_0 + \sum_{k=1}^{N} c_k z^{n_k}\right) \tag{A.9}$$

or

$$f(z) = \cos\left(c_0 + \sum_{k=1}^{N} c_k z^{n_k}\right) \tag{A.10}$$

It is obvious that the functions (A.9) and (A.10) have no singular points.

Lemma 2 The function (A.9) satisfies Cauchy-Riemann conditions throughout the definition.

Proof. First let's present the function (3) in the form $f(z) = u(x,y) + iv(x,y)$:

$$\begin{aligned} f(z) &= F\left(c_0 + \sum_{k=1}^{N} c_k z^{n_k}\right) = F\left(c_0 + \sum_{k=1}^{N} c_k (x+iy)^{n_k}\right) = \\ &= F\left(c_0 + \sum_{k=1}^{N} c_k \sum_{l=0}^{n_k} C_{n_k}^{l} x^{n_k - l} (iy)^l\right) = \\ &= F\Bigg(c_0 + \sum_{k=1}^{N} c_k \sum_{l=0}^{[n_k/2]} C_{n_k}^{2l} x^{n_k-2l}(-1)^l y^{2l} \\ &\quad + i\sum_{k=1}^{N} c_k \sum_{l=0}^{[(n_k-1)/2]} C_{n_k}^{2l+1} x^{n_k-2l-1}(-1)^l y^{2l+1}\Bigg) = \\ &= F\left(Re + iIm\right) \end{aligned} \tag{A.11}$$

Here $Re(x,y)$, $Im(x,y)$ are defined by (A.6). Calculate Re'_x, Im'_y, Re'_y, Im'_x.

$$Re'_x = \frac{\partial Re}{\partial x} = \sum_{k=1}^{N} c_k \sum_{l=0}^{[(n_k-1)/2]} C_{n_k}^{2l}(n_k - 2l) x^{n_k-2l-1}(-1)^l y^{2l};$$

$$Re'_y = \frac{\partial Re}{\partial y} = \sum_{k=1}^{N} c_k \sum_{l=0}^{[n_k/2]} C_{n_k}^{2l} x^{n_k-2l}(-1)^l (2l) y^{2l-1};$$

$$Im'_x = \frac{\partial Im}{\partial x} = \sum_{k=1}^{N} c_k \sum_{l=0}^{[(n_k-1)/2]} C_{n_k}^{2l+1}(n_k - 2l - 1) x^{n_k-2l-2}(-1)^l y^{2l+1};$$

$$Im'_y = \frac{\partial Im}{\partial y} = \sum_{k=1}^{N} c_k \sum_{l=0}^{[(n_k-1)/2]} C_{n_k}^{2l+1} x^{n_k-2l-1}(-1)^l (2l+1) y^{2l}.$$

Calculate the following differences:

$$Re'_x - Im'_y = \sum_{k=1}^{N} c_k \Biggl(\sum_{l=0}^{[(n_k-1)/2]} C_{n_k}^{2l}(n_k-2l)x^{n_k-2l-1}(-1)^l y^{2l} -$$
$$- \sum_{l=0}^{[(n_k-1)/2]} C_{n_k}^{2l+1} x^{n_k-2l-1}(-1)^l(2l+1)y^{2l} \Biggr) =$$
$$= \sum_{k=1}^{N} c_k \Biggl(\sum_{l=0}^{[(n_k-1)/2]} x^{n_k-2l-1}(-1)^l y^{2l} \Biggl(\frac{n_k!}{(2l)!(n_k-2l)!}(n_k-2l) -$$
$$- \frac{n_k!}{(2l+1)!(n_k-2l-1)!}(2l+1) \Biggr)\Biggr) = 0;$$
$$Re'_y + Im'_x = \sum_{k=1}^{N} c_k \Biggl(\sum_{l=0}^{[n_k/2]} C_{n_k}^{2l} x^{n_k-2l}(-1)^l(2l)y^{2l-1} +$$
$$+ \sum_{l=0}^{[(n_k-1)/2]} C_{n_k}^{2l+1}(n_k-2l-1)x^{n_k-2l-2}(-1)^l y^{2l+1} \Biggr) =$$
$$= \sum_{k=1}^{N} c_k \Biggl(\sum_{l=0}^{[n_k/2]} \frac{n_k!}{(2l)!(n_k-2l)!}(2l)x^{n_k-2l}(-1)^l y^{2l-1} -$$
$$- \sum_{l=0}^{[n_k/2]} \frac{n_k!}{(2l-1)!(n_k-2l+1)!}(n_k-2l+1)x^{n_k-2l}(-1)^l y^{2l-1} \Biggr) = 0.$$

So, it is derived that

$$Re'_x = Im'_y, Re'_y = -Im'_x \qquad (A.12)$$

takes place.

Using (A.11), the properties of trigonometric functions and Euler formulae the function (A.9) can be rewritten as $f(z) = \sin(Re + iIm) = \sin Re \cos(iIm) + \cos Re \sin(iIm) = \sin Re \cosh Im + i \cos Re \sinh Im = u(x,y) + iv(x,y)$, where $u(x,y) = \sin Re \cosh Im, v(x,y) = \cos Re \sinh Im$.

Calculate the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$:

$$\frac{\partial u}{\partial x} = \cos Re Re'_x \cosh Im + \sin Re \sinh Im Im'_x;$$
$$\frac{\partial v}{\partial y} = -\sin Re Re'_y \sinh Im + \cos Re \cosh Im Im'_y;$$

$$\frac{\partial u}{\partial y} = \cos Re Re'_y \cosh Im + \sin Re \sinh Im Im'_y;$$
$$\frac{\partial v}{\partial x} = -\sin Re Re'_x \sinh Im + \cos Re \cosh Im Im'_x.$$

Using (A.12), it is derived that Cauchy-Riemann conditions (A.3) are fulfilled for the function (A.9) for all z.

Lemma 3 The function (A.10) satisfies Cauchy-Riemann conditions throughout the definition.

Proof. Using (A.11), the properties of trigonometric functions and Euler formulae the function (A.10) can be rewritten as $f(z) = \cos(Re + iIm) = \cos Re \cos(iIm) - \sin Re \sin(iIm) = \cos Re \cosh Im - i \sin Re \sinh Im = u(x,y) + iv(x,y)$, where $u(x,y) = \cos Re \cosh Im, v(x,y) = -\sin Re \sinh Im$.

Calculate the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$:

$$\frac{\partial u}{\partial x} = -\sin Re Re'_x \cosh Im + \cos Re \sinh Im Im'_x;$$
$$\frac{\partial v}{\partial y} = -\cos Re Re'_y \sinh Im - \sin Re \cosh Im Im'_y;$$
$$\frac{\partial u}{\partial y} = -\sin Re Re'_y \cosh Im + \cos Re \sinh Im Im'_y;$$
$$\frac{\partial v}{\partial x} = -\cos Re Re'_x \sinh Im - \sin Re \cosh Im Im'_x.$$

Using (A.12), it is derived that Cauchy-Riemann conditions (A.3) are fulfilled for the function (A.10) for all z.

A.3. Hyperbolic case

The transform (2) can be written in the following form

$$\sinh\left(c_0 + \sum_{i=1}^{N} c_i e^{-sn_i A_q}\right) \qquad (A.13)$$

or

$$\cosh\left(c_0 + \sum_{i=1}^{N} c_i e^{-sn_i A_q}\right) \qquad (A.14)$$

The function (3) in this case can be written as

$$f(z) = \sinh\left(c_0 + \sum_{k=1}^{N} c_k z^{n_k}\right) \qquad (A.15)$$

or

$$f(z) = \cosh\left(c_0 + \sum_{k=1}^{N} c_k z^{n_k}\right) \qquad (A.16)$$

It is obvious that the functions (A.15) and (A.16) have no singular points.

Lemma 4 The function (A.15) satisfies Cauchy-Riemann conditions throughout the definition.

Proof. Using (A.11), the properties of hyperbolic functions and Euler formulae the function (A.15) can be rewritten as $f(z) = \sinh(Re + iIm) = \sinh Re \cosh(iIm) + \cosh Re \sinh(iIm) = \sinh Re \cos Im + i \cosh Re \sin Im = u(x,y) + iv(x,y)$, where $u(x,y) = \sinh Re \cos Im, v(x,y) = \cosh Re \sin Im$.

Calculate the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$:

$$\frac{\partial u}{\partial x} = \cosh Re Re'_x \cos Im - \sinh Re \sin Im Im'_x;$$
$$\frac{\partial v}{\partial y} = \sinh Re Re'_y \sin Im + \cosh Re \cos Im Im'_y;$$
$$\frac{\partial u}{\partial y} = \cosh Re Re'_y \cos Im - \sinh Re \sin Im Im'_y;$$
$$\frac{\partial v}{\partial x} = \sinh Re Re'_x \sin Im + \cosh Re \cos Im Im'_x.$$

Using (A.12), it is derived that Cauchy-Riemann conditions (A.3) are fulfilled for the function (A.15) for all z. □

Lemma 5 The function (A.16) satisfies Cauchy-Riemann conditions throughout the definition.

Proof. Using (A.11), the properties of hyperbolic functions and Euler formulae the function (A.10) can be rewritten as $f(z) = \cosh(Re + iIm) = \cosh Re \cosh(iIm) - \sinh Re \sinh(iIm) = \cosh Re \cos Im + i \sinh Re \sin Im = u(x,y) + iv(x,y)$, where $u(x,y) = \cosh Re \cos Im, v(x,y) = \sinh Re \sin Im$.

Calculate the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$:

$$\frac{\partial u}{\partial x} = \sinh Re Re'_x \cos Im - \cosh Re \sin Im Im'_x;$$
$$\frac{\partial v}{\partial y} = \cosh Re Re'_y \sin Im + \sinh Re \cos Im Im'_y;$$
$$\frac{\partial u}{\partial y} = \sinh Re Re'_y \cos Im - \cosh Re \sin Im Im'_y;$$

$$\frac{\partial v}{\partial x} = \cosh Re Re'_x \sin Im + \sinh Re \cos Im Im'_x.$$

Using (A.12), it is derived that Cauchy-Riemann conditions (A.3) are fulfilled for the function (A.16) for all z.

A.4. Exponential case

The transform (2) can be written in the following form

$$\exp\left(c_0 + \sum_{i=1}^{N} c_i e^{-sn_i A_q}\right) \qquad (A.17)$$

The function (3) in this case can be written as

$$f(z) = \exp\left(c_0 + \sum_{k=1}^{N} c_k z^{n_k}\right) \qquad (A.18)$$

It is obvious that the function (A.18) has no singular points.

Lemma 6 The function (A.18) satisfies Cauchy-Riemann conditions throughout the definition.

Proof. Using (A.11) and Euler formulae the function (A.18) can be rewritten as $f(z) = e^{Re}\left(\cos Im + i \sin Im\right) = e^{Re} \cos Im + i e^{Re} \sin Im = u(x,y) + iv(x,y)$, where $u(x,y) = e^{Re} \cos Im, v(x,y) = e^{Re} \sin Im$.

Calculate the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= e^{Re} Re'_x \cos Im - e^{Re} \sin Im Im'_x;\\
\frac{\partial v}{\partial y} &= e^{Re} Re'_y \sin Im + e^{Re} \cos Im Im'_y;\\
\frac{\partial u}{\partial y} &= e^{Re} Re'_y \cos Im - e^{Re} \sin Im Im'_y;\\
\frac{\partial v}{\partial x} &= e^{Re} Re'_x \sin Im + e^{Re} \cos Im Im'_x.
\end{aligned}$$

Using (A.12), it is derived that Cauchy-Riemann conditions (A.3) are fulfilled for the function (A.18) for all z.

B. Examples and verification

B.1. Verification with the previously known results

The verification of the proposed method is done on the known transforms. Consider the functions $\frac{1}{1-e^{-sA}}$ and $\frac{1}{1+e^{-sA}}$ when $A > 0$. From [24] it is known that

$$L^{-1}\left[\frac{1}{1-e^{-sA}}\right] = \sum_{n=0}^{\infty} \delta(t - nA), \qquad (B.1)$$

$$L^{-1}\left[\frac{1}{1+e^{-sA}}\right] = \sum_{n=0}^{\infty} (-1)^n \delta(t - nA) \qquad (B.2)$$

Let's show that the results derived from theorem 1 are consistent with the known results (B.1)-(B.2).

According to theorem 1

$$L^{-1}\left[\frac{1}{1-e^{-sA}}\right] = \left[z = e^{-sA}\right] = L^{-1}\left[\frac{1}{1-z}\right] = \sum_{k=0}^{\infty} \delta(t - kA), \qquad (B.3)$$

which is congruent to (B.1).

$$L^{-1}\left[\frac{1}{1+e^{-sA}}\right] = \left[z = e^{-sA}\right] = L^{-1}\left[\frac{1}{1+z}\right] = \sum_{k=0}^{\infty} (-1)^k \delta(t - kA), \quad (B.4)$$

which is congruent to (B.2).

So, the known results (B.1)-(B.2) are equal to the results derived from theorem 1 (B.3)-(B.4).

Let's consider some examples of application of the proved theorems.

B.2. Some examples based on the theorem 1

Example 1 Consider the following functions $\frac{1}{(1-de^{-sA})^{\alpha}}$ and $\frac{1}{(1+de^{-sA})^{\alpha}}$ when $A, d > 0$ are some digits, α is a natural digit.

The Taylor series can be easily constructed for the functions $f(z) = \frac{1}{(1-dz)^{\alpha}}$ and $g(z) = \frac{1}{(1+dz)^{\alpha}}$:

$$f(z) = 1 + \sum_{k=1}^{\infty} \frac{d^k \alpha(\alpha+1)...(\alpha+k-1)}{k!} z^k$$

$$g(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k d^k \alpha(\alpha+1)...(\alpha+k-1)}{k!} z^k$$

According to theorem 1

$$L^{-1}\left[\frac{1}{(1-de^{-sA})^{\alpha}}\right]=\left[z=e^{-sA}\right]=L^{-1}\left[f(z)\right]=$$
$$=\delta(t)+\sum_{k=1}^{\infty}\frac{d^{k}\alpha(\alpha+1)...(\alpha+k-1)}{k!}\delta(t-kA)$$
$$L^{-1}\left[\frac{1}{(1+de^{-sA})^{\alpha}}\right]=\left[z=e^{-sA}\right]=L^{-1}\left[g(z)\right]=$$
$$=\delta(t)+\sum_{k=1}^{\infty}\frac{(-1)^{k}d^{k}\alpha(\alpha+1)...(\alpha+k-1)}{k!}\delta(t-kA).$$

Finally the following formulas are derived

$$L^{-1}\left[\frac{1}{(1-de^{-sA})^{\alpha}}\right]=\delta(t)+\sum_{k=1}^{\infty}\frac{d^{k}\alpha(\alpha+1)...(\alpha+k-1)}{k!}\delta(t-kA)\quad (B.5)$$

$$L^{-1}\left[\frac{1}{(1+de^{-sA})^{\alpha}}\right]=\delta(t)+\sum_{k=1}^{\infty}\frac{(-1)^{k}d^{k}\alpha(\alpha+1)...(\alpha+k-1)}{k!}\delta(t-kA)\quad (B.6)$$

Let's verify the derived formulas (B.5)-(B.6) with the use of convolution. It can be done for any fixed α and any $d>0$. Let's prove this for $\alpha=2$.

According to (B.5), (B.6)

$$L^{-1}\left[\frac{1}{(1-de^{-sA})^{2}}\right]=\delta(t)+\sum_{k=1}^{\infty}d^{k}(k+1)\delta(t-kA),\quad (B.7)$$

$$L^{-1}\left[\frac{1}{(1+de^{-sA})^{2}}\right]=\delta(t)+\sum_{k=1}^{\infty}(-1)^{k}d^{k}(k+1)\delta(t-kA)\quad (B.8)$$

Consider the following convolution

$$\left(\left[\delta(t)-2d\delta(t-A)+d^{2}\delta(t-2A)\right]*\left[\delta(t)+\sum_{k=1}^{\infty}d^{k}(k+1)\delta(t-kA)\right],\varphi(t)\right)=$$
$$=\iint\limits_{\mathbb{R}^2}\left[\delta(\xi)-2d\delta(\xi-A)+d^{2}\delta(\xi-2A)\right]\times$$
$$\times\left[\delta(x-\xi)+\sum_{k=1}^{\infty}d^{k}(k+1)\delta(x-\xi-kA)\right]\varphi(x)dxd\xi=$$
$$=\varphi(0)-2d\varphi(A)+d^{2}\varphi(2A)+\sum_{k=1}^{\infty}d^{k}(k+1)\varphi(kA)-$$

$$-2\sum_{k=2}^{\infty} d^k k\varphi(kA) + \sum_{k=3}^{\infty} d^k (k-1)\varphi(kA) = \varphi(0) = (\delta(t), \varphi(t))$$

So, it is proved that

$$\left[\delta(t) - 2d\delta(t-A) + d^2\delta(t-2A)\right] * \left[\delta(t) + \sum_{k=1}^{\infty} d^k (k+1)\delta(t-kA)\right] = \delta(t).$$

The equality

$$\left[\delta(t) + \sum_{k=1}^{\infty} d^k (k+1)\delta(t-kA)\right] * \left[\delta(t) - 2d\delta(t-A) + d^2\delta(t-2A)\right] = \delta(t)$$

is proved similarly. So, the correctness of the formula (B.7) is shown.

Consider the following convolution

$$\left(\left[\delta(t) + 2d\delta(t-A) + d^2\delta(t-2A)\right] * \left[\delta(t) + \sum_{k=1}^{\infty} (-1)^k d^k (k+1)\delta(t-kA)\right], \varphi(t)\right) =$$

$$= \iint\limits_{\mathbb{R}^2} \left[\delta(\xi) + 2d\delta(\xi-A) + d^2\delta(\xi-2A)\right] \times$$

$$\times \left[\delta(x-\xi) + \sum_{k=1}^{\infty} (-1)^k d^k (k+1)\delta(x-\xi-kA)\right] \varphi(x)dxd\xi =$$

$$= \varphi(0) + 2d\varphi(A) + d^2\varphi(2A) + \sum_{k=1}^{\infty} (-1)^k d^k (k+1)\varphi(kA) -$$

$$-2\sum_{k=2}^{\infty} (-1)^k d^k k\varphi(kA) + \sum_{k=3}^{\infty} (-1)^k d^k (k-1)\varphi(kA) = \varphi(0) = (\delta(t), \varphi(t)).$$

So, it is proved that

$$\left[\delta(t) + 2d\delta(t-A) + d^2\delta(t-2A)\right] * \left[\delta(t) + \sum_{k=1}^{\infty} (-1)^k d^k (k+1)\delta(t-kA)\right] = \delta(t).$$

The equality

$$\left[\delta(t) + \sum_{k=1}^{\infty} (-1)^k d^k (k+1)\delta(t-kA)\right] * \left[\delta(t) + 2d\delta(t-A) + d^2\delta(t-2A)\right] = \delta(t)$$

is proved similarly. So, the correctness of the formula (B.8) is shown.

Example 2 Consider the following functions $\ln\left|1 - de^{-sA}\right|$ and $\ln\left(1 + de^{-sA}\right)$ when $A, d > 0$ are some digits.

The Taylor series can be easily constructed for the functions $f(z) = \ln|1 - dz|$ and $g(z) = \ln(1 + dz)$:

$$f(z) = -\sum_{k=1}^{\infty} \frac{d^k}{k} z^k$$
$$g(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} d^k}{k} z^k$$

According to theorem 1

$$L^{-1}\left[\ln\left|1 - de^{-sA}\right|\right] = -\sum_{k=1}^{\infty} \frac{d^k}{k}\delta(t - kA)$$
$$L^{-1}\left[\ln\left(1 + de^{-sA}\right)\right] = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}d^k}{k}\delta(t - kA)$$

B.3. Some examples based on the theorem 2

Example 1 Consider the function $\frac{1}{1-pe^{-sA}-qe^{-sB}}$, where $A, B, p, q > 0, A \neq B$. After the change of the variables $z_1 = e^{-sA}, z_2 = e^{-sB}$ the initial function can be rewritten as $f(z_1, z_2) = \frac{1}{1-pz_1-qz_2}$. The Taylor series can be easily constructed for this function: $f(z_1, z_2) = \frac{1}{1-pz_1-qz_2} = \sum\limits_{i=0}^{\infty}\sum\limits_{j=0}^{\infty} C_{i+j}^i p^i q^j z_1^i z_2^j$, where $C_{i+j}^i = \frac{(i+j)!}{i!j!}$ are binomial coefficients.

According to theorem 2

$$L^{-1}\left[\frac{1}{1 - pe^{-sA} - qe^{-sB}}\right] = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} C_{i+j}^i p^i q^j \delta(t - iA - jB) \qquad (B.9)$$

Consider the following convolution

$$\left([\delta(t) - p\delta(t - A) - q\delta(t - B)] * \left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} C_{i+j}^i p^i q^j \delta(t - iA - jB)\right], \varphi(t)\right) =$$
$$= \iint_{R^2} [\delta(\xi) - p\delta(\xi - A) - q\delta(\xi - B)] \times$$
$$\times \left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} C_{i+j}^i p^i q^j \delta(x - \xi - iA - jB)\right] \varphi(x)dxd\xi =$$
$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(i+j)!}{i!j!} p^i q^j \varphi(iA + jB) - \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \frac{(i+j-1)!}{(i-1)!j!} p^i q^j \varphi(iA + jB) -$$

$$-\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\frac{(i+j-1)!}{i!(j-1)!}p^i q^j \varphi(iA+jB)=\varphi(0)=(\delta(t),\varphi(t))$$

So, it is proved that

$$[\delta(t)-p\delta(t-A)-q\delta(t-B)]*\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}C_{i+j}^{i}p^i q^j\delta(t-iA-jB)\right]=\delta(t).$$

The equality

$$\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}C_{i+j}^{i}p^i q^j\delta(t-iA-jB)\right]*[\delta(t)-p\delta(t-A)-q\delta(t-B)]=\delta(t)$$

is proved similarly. So, the correctness of the formula (B.9) is shown.

Let's prove that when $A=B>0$ the inverse formula (B.9) is congruent to the inverse formula (B.5) for the case 1.

When $A=B>0$ $\frac{1}{1-pe^{-sA}-qe^{-sB}}=\frac{1}{1-(p+q)e^{-sA}}$. According to (B.5) when $d=p+q, \alpha=1$

$$L^{-1}\left[\frac{1}{1-(p+q)e^{-sA}}\right]=\sum_{k=0}^{\infty}(p+q)^k\delta(t-kA) \qquad (B.10)$$

Let's show that the expression (B.9) coincides with (B.10) in the case when $A=B$. We have

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}C_{i+j}^{i}p^i q^j\delta(t-(i+j)A)=[k=i+j]=$$

$$=\sum_{k=0}^{\infty}\delta(t-kA)\sum_{j=0}^{k}C_k^j p^{k-j}q^j=\sum_{k=0}^{\infty}(p+q)^k\delta(t-kA),$$

which coincides with (B.10).

Example 2 Consider the function $\frac{1}{1+pe^{-sA}+qe^{-sB}}$, where $A,B,p,q>0, A\neq B$. After the change of the variables $z_1=e^{-sA}, z_2=e^{-sB}$ the initial function can be rewritten as $f(z_1,z_2)=\frac{1}{1+pz_1+qz_2}$. The Taylor series can be easily constructed for this function: $f(z_1,z_2)=\frac{1}{1+pz_1+qz_2}=\sum\limits_{i=0}^{\infty}\sum\limits_{j=0}^{\infty}(-1)^{i+j}p^i q^j C_{i+j}^{i}z_1^i z_2^j$.

According to theorem 2

$$L^{-1}\left[\frac{1}{1+pe^{-sA}+qe^{-sB}}\right]=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}p^i q^j C_{i+j}^{i}\delta(t-iA-jB) \quad (B.11)$$

Consider the following convolution

$$\left([\delta(t)+p\delta(t-A)+q\delta(t-B)]*\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}p^iq^jC_{i+j}^i\delta(t-iA-jB)\right],\varphi(t)\right)=$$

$$=\iint\limits_{R^2}[\delta(\xi)+p\delta(\xi-A)+q\delta(\xi-B)]\times$$

$$\times\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}C_{i+j}^ip^iq^j\delta(x-\xi-iA-jB)\right]\varphi(x)dxd\xi=$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}p^iq^j\frac{(i+j)!}{i!j!}\varphi(iA+jB)-$$

$$-\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\frac{(i+j-1)!}{(i-1)!j!}p^iq^j\varphi(iA+jB)-$$

$$-\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}(-1)^{i+j}\frac{(i+j-1)!}{i!(j-1)!}p^iq^j\varphi(iA+jB)=\varphi(0)=(\delta(t),\varphi(t))$$

So, it is proved that

$$[\delta(t)+p\delta(t-A)+q\delta(t-B)]*\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}C_{i+j}^ip^iq^j\delta(t-iA-jB)\right]=\delta(t).$$

The equality

$$\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}C_{i+j}^ip^iq^j\delta(t-iA-jB)\right]*[\delta(t)+p\delta(t-A)+q\delta(t-B)]=\delta(t)$$

is proved similarly. So, the correctness of the formula (B.11) is shown.

Let's prove that when $A=B>0$ the inverse formula (B.11) is congruent to the inverse formula (B.6) for the case 1.

When $A=B>0$ $\frac{1}{1+pe^{-sA}+qe^{-sB}}=\frac{1}{1+(p+q)e^{-sA}}$. According to (B.6) when $d=p+q,\alpha=1$

$$L^{-1}\left[\frac{1}{1+(p+q)e^{-sA}}\right]=\sum_{k=0}^{\infty}(-1)^k(p+q)^k\delta(t-kA)\qquad(B.12)$$

Let's show that the expression (B.11) coincides with (B.12) in the case when

$A = B$. $\sum\limits_{i=0}^{\infty}\sum\limits_{j=0}^{\infty}(-1)^{i+j}C_{i+j}^{i}p^{i}q^{j}\delta(t-(i+j)A) = [k = i+j] = \sum\limits_{k=0}^{\infty}(-1)^{k}\delta(t-kA)\sum\limits_{j=0}^{k}C_{k}^{j}p^{k-j}q^{j} = \sum\limits_{k=0}^{\infty}(-1)^{k}(p+q)^{k}\delta(t-kA)$, which coincides with (B.12).

Analogically to the examples 1-2 the inverse formulas for the following functions can be written:

$$L^{-1}\left[\frac{1}{1-pe^{-sA}+qe^{-sB}}\right] = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{j}C_{i+j}^{i}p^{i}q^{j}\delta(t-iA-jB)$$

$$L^{-1}\left[\frac{1}{1+pe^{-sA}-qe^{-sB}}\right] = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i}C_{i+j}^{i}p^{i}q^{j}\delta(t-iA-jB)$$

Example 3 Consider the more general functions $\frac{1}{(1-pe^{-sA}-qe^{-sB})^{\alpha}}$ and $\frac{1}{(1+pe^{-sA}+qe^{-sB})^{\alpha}}$ when $A, B, p, q > 0, A \neq B$, α is a natural digit. After the change of the variables $z_1 = e^{-sA}, z_2 = e^{-sB}$ the initial functions can be rewritten as $f(z_1, z_2) = \frac{1}{(1-pz_1-qz_2)^{\alpha}}, g(z_1, z_2) = \frac{1}{(1+pz_1+qz_2)^{\alpha}}$. The Taylor series can be easily constructed for these functions:

$$f(z_1, z_2) = \frac{1}{(1-pz_1-qz_2)^{\alpha}} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}p^{i}q^{j}\frac{\psi_{i+j}(\alpha)}{i!j!}z_1^{i}z_2^{j},$$

$$g(z_1, z_2) = \frac{1}{(1+pz_1+qz_2)^{\alpha}} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}p^{i}q^{j}\frac{\psi_{i+j}(\alpha)}{i!j!}z_1^{i}z_2^{j}.$$

Here $\psi_n(\alpha) = \alpha(\alpha+1)...(\alpha+n-1) = (\alpha)_n$ when $n > 0$ and $\psi_0(\alpha) = 1$.

According to theorem 2

$$L^{-1}\left[\frac{1}{(1-pe^{-sA}-qe^{-sB})^{\alpha}}\right] = \left[z = e^{-sA}\right] = L^{-1}\left[f(z_1, z_2)\right] = \\ = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}p^{i}q^{j}\frac{\psi_{i+j}(\alpha)}{i!j!}\delta(t-iA-jB),$$

$$L^{-1}\left[\frac{1}{(1+pe^{-sA}+qe^{-sB})^{\alpha}}\right] = \left[z = e^{-sA}\right] = L^{-1}\left[g(z_1, z_2)\right] = \\ = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}p^{i}q^{j}\frac{\psi_{i+j}(\alpha)}{i!j!}\delta(t-iA-jB).$$

Finally the following formulas are derived

$$L^{-1}\left[\frac{1}{(1-pe^{-sA}-qe^{-sB})^{\alpha}}\right] = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}p^{i}q^{j}\frac{\psi_{i+j}(\alpha)}{i!j!}\delta(t-iA-jB) \quad (B.13)$$

$$L^{-1}\left[\frac{1}{(1+pe^{-sA}+qe^{-sB})^{\alpha}}\right]=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}p^{i}q^{j}\frac{\psi_{i+j}(\alpha)}{i!j!}\delta(t-iA-jB) \qquad (B.14)$$

Analogically to (B.13)–(B.14) the inverse formulas for the following functions can be written:

$$L^{-1}\left[\frac{1}{(1-pe^{-sA}+qe^{-sB})^{\alpha}}\right]=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{j}p^{i}q^{j}\frac{\psi_{i+j}(\alpha)}{i!j!}\delta(t-iA-jB)$$

$$L^{-1}\left[\frac{1}{(1+pe^{-sA}-qe^{-sB})^{\alpha}}\right]=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i}p^{i}q^{j}\frac{\psi_{i+j}(\alpha)}{i!j!}\delta(t-iA-jB)$$

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Журавльова З. Ю.
Випадок аналітичного обернення перетворення Лапласа

Резюме

У даній статті запропоновано новий метод аналітичного обернення перетворення Лапласу для трансформант, що містять експоненти, які лінійно залежать від параметра перетворення Лапласу. Даний метод заснований на розвиненні трансформанти у ряд Тейлора и почленному застосуванні оберненого перетворення Лапласу. Доведено теореми, що підтверждують достовірність та коректність такого підходу. Цей метод використовує узагальнені функції, тому отримано деякі корисні наслідки, що пов'язані з

узагальненими функціями. Метод перевірений шляхом порівняння з відомими з літератури формулами. Отримані нові формули для оригиналів від трансформант Лапласу.
Ключові слова: перетворення Лапласу, аналітичне обернення, ряди Тейлора, узагальнені функції, згортка.

Журавлёва З. Ю.

Случай аналитического обращения преобразования Лапласа

Резюме

В данной статье предложен новый метод аналитического обращения преобразования Лапласа для трансформант, которые содержат экспоненты, линейно зависящие от параметра преобразования Лапласа. Данный метод основан на разложении трансформанты в ряд Тейлора и почленном применении обратного преобразования Лапласа. Доказаны теоремы, подтверждающие достоверность и корректность такого подхода. Этот метод использует обобщённые функции, поэтому получены некоторые полезные следствия, связанные с обратными обобщёнными функциями. Метод проверен путём сравнения с известными из литературы формулами. Получены новые формулы для оригиналов от трансформант Лапласа.
Ключевые слова: преобразование Лапласа, аналитическое обращение, ряды Тейлора, обобщённые функции, свёртка.

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