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GAUSSIAN INTEGERS PARTITION IN POWER-FREE NUMBERS PRODUCT

Let $g_1(\alpha)$ be the number of Gaussian integer α representation in a product of square-free factors. Let $g_2(\alpha)$ be the number of Gaussian integer α representation in a product of power-free factors. In this paper we consider their summatory functions $\sum_{N(\alpha)\leq x} g_1(\alpha)$ and $\sum_{N(\alpha)\leq x} g_2(\alpha)$ and obtain asymptotic formulas for them. Also, we prove analogue of Kátai-Subbarao theorem to study the distribution of $g_2(\alpha)$ in increasing norm order case.

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Key words: Hecke zetafunction; square-free Gaussian integer; power-free Gaussian integer; Dirichlet generating series.

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1. Introduction

Let G denote a ring of Gaussian integers

$$G{=}\mathbb{Z}\left[i\right]=\left\{a+ib \mid a,b\in\mathbb{Z},\, i^2=-1\right\}.$$

Let $\mathfrak{p}$ denote a Gaussian prime integer.

Let Gaussian integer α be power-free if $\alpha=\mathfrak{p}_1^{k_1}\mathfrak{p}_2^{k_2}\cdots\mathfrak{p}_r^{k_r}$ and

$$GCD\left(k_1,k_2,\ldots,k_r\right)=1,$$

where $k_i\in\mathbb{N}$, $i=\overline{1;r}$. In other words, α is power-free if there is no Gaussian integer β such that $\alpha=\beta^k$, $k\in\{2,\ 3,\ \ldots\}$. Let us notice that all square-free numbers are power-free.

Let Gaussian integer α be square-free if for any Gaussian prime integer $\mathfrak{p}$ such that $\mathfrak{p}\mid\alpha$ there is no positive integer $k>1$ that $\mathfrak{p}^k$ divides α. Also notice that all square-free numbers are power-free.

Each Gaussian integer α can be represented as the product of power-free (square-free) numbers except $\varepsilon\in\{\pm1,\ \pm i\}$. Therefore let $g_2\left(\alpha\right)$ $\left(g_1\left(\alpha\right)\right)$ denote the number of Gaussian integer α representation in a product of power-free $\left(\text{square-free}\right)$ numbers.

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For example, consider the following representations of $\alpha = \mathfrak{p}_1^2 \cdot \mathfrak{p}_2^3$ in a product of power-free factors

$$\alpha = \mathfrak{p}_1^2 \cdot \mathfrak{p}_2^3 = \mathfrak{p}_1 \cdot \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_2 = \mathfrak{p}_1\mathfrak{p}_2 \cdot \mathfrak{p}_1\mathfrak{p}_2 \cdot \mathfrak{p}_2$$
$$= \mathfrak{p}_1 \cdot \mathfrak{p}_1\mathfrak{p}_2 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_2 = \mathfrak{p}_1 \cdot \left(\mathfrak{p}_1\mathfrak{p}_2^3\right) = \mathfrak{p}_1 \cdot \left(\mathfrak{p}_1\mathfrak{p}_2^2\right) \cdot \mathfrak{p}_2 = \mathfrak{p}_1\mathfrak{p}_2 \cdot \left(\mathfrak{p}_1\mathfrak{p}_2^2\right).$$

$\mathrm{g}_2(\alpha) = 7.$

In case of square-free factors α has the following representations

$$\alpha = \mathfrak{p}_1 \cdot \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_2 = \mathfrak{p}_1\mathfrak{p}_2 \cdot \mathfrak{p}_1\mathfrak{p}_2 \cdot \mathfrak{p}_2 = \mathfrak{p}_1 \cdot \mathfrak{p}_1\mathfrak{p}_2 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_2.$$

$\mathrm{g}_1(\alpha) = 3.$

By $\mathrm{g}_2^*(\alpha)$ we denote the function below

$$\mathrm{g}_2^*(\alpha) = \#\left\{\alpha = \delta_1\delta_2\ldots\delta_r \;\middle|\; \delta_i \text{ are power-free, } i = \overline{1;r},\right.$$
$$\left. N(\delta_1) \leq N(\delta_2) \leq \cdots \leq N(\delta_r)\right\},$$

where $N(\alpha)$ is the norm of α (i.e. if $\alpha = \sigma + it$, then $N(\alpha) = \sigma^2 + t^2$).

The purpose of this paper is to prove the asymptotic formula for the summatory functions of $\mathrm{g}_1(\alpha)$, $\mathrm{g}_2(\alpha)$ and $\mathrm{g}_2^*(\alpha)$. These are a generalization of the results of A. Korchevskiy and Ya. Vorobyov in positive integer case.

2. Auxiliary results

Let us consider Hecke zetafunction $Z_m(s)$ with the Hecke character $\lambda_m(\alpha)$

$$Z_m(s) = \sum_{0 \neq \alpha \in G} \frac{\lambda_m(\alpha)}{N(\alpha)^s},$$

where $\lambda_m(\alpha) = \exp(mi \arg \alpha)$, α is a Gaussian integer, $m \in \mathbb{Z}$, Re $s > 1$.

Moreover, we are interested in the case when $m = 4m_1,\ m_1 \in \mathbb{Z}$ for $\lambda_m(\alpha)$ be the same for associated Gaussian integers.

Thus, for associated Gaussian integers α and $\varepsilon\alpha$, where $\varepsilon \in \{\pm 1,\ \pm i\}$, the following relation

$$\lambda_m(\alpha) = \lambda_m(\varepsilon\alpha)$$

holds, because

$$\exp(4mi \arg \alpha) = \exp(4mi \arg \varepsilon\alpha).$$

The function $Z_m(s)$ can be analytically continuated to the entire s-plane, except the point $s = 1$, where it has a simple pole with residue π.

For $m = 0$ we have $Z_0(s) = 4\,\zeta(s)\,L\big(s, \chi_4(n)\big)$, where $\zeta(s)$ is Riemann zetafunction, $\chi_4(n)$ is non-main Dirichlet character modulo 4.

Hecke zetafunction $Z_m(s)$ satisfies the functional equation

$$Z_m(s) = \pi^{\,2s-1} \frac{\Gamma(\frac{m}{2} + 1 - s)}{\Gamma(\frac{m}{2} + s)}\, Z_m(1 - s),$$

where Re $s > \frac{1}{2}$, $\Gamma(s) = \int\limits_0^\infty t^{s-1} \exp(-t) dt$ is the gamma function.

It is known that the absolute value of a regular function in the interior of a bounded region is bounded by its absolute value on the boundary of the region.

Moving the function to the left of the line Re $s = \frac{1}{2}$ and using Phragm?n-Lindelöf principle we can get the following estimates for Hecke zetafunction in critical strip.

Lemma 1 (Estimates for Hecke zetafunction in critical strip)**.**

$$Z_m\,(\sigma + it) \ll \begin{cases} 1, & \textit{if}\;\; \sigma \geq 1 + \varepsilon, \\ \log\left|t^2 + m^2\right|, & \textit{if}\;\; 1 \leq \sigma \leq 1 + \varepsilon, \\ \left(t^2 + m^2\right)^{\frac{1-\sigma}{2}} \log\left(1 + \varepsilon\;\right), & \textit{if}\;\; 0 \leq \sigma < 1. \end{cases}$$

Our purpose is to study special arithmetic functions $\mathrm{g}_1\,(\alpha)$, $\mathrm{g}_2\,(\alpha)$ and $\mathrm{g}_2^*\,(\alpha)$. These functions are related to the functions studied by Kátai and Subbarao in [2].

Let $e(n)$ be an arbitrary arithmetic function. We will assume that $e(n) \geq 0$, $e(n) \ll n^{\varepsilon}$, where $\varepsilon > 0$ is arbitrarily small.

Theorem 1 (Kátai-Subbarao [2], Theorem 5.1)**.** *Let* $\{e\,(n)\}$ *and* $\{f\,(n)\}$ *be sequences that satisfy the relation*

$$\prod_{n\,=\,2}^{\infty} \left(1 + \frac{e\,(n)}{n^s}\right) = \sum_{n\,=\,1}^{\infty} \frac{f\,(n)}{n^s}\;.$$

Moreover $e\,(1) = f\,(1) = 1$. Since we chose the function $e(n)$ so that $e(n) \geq 0$ and $e(n) \ll n^{\varepsilon}$, where $\varepsilon > 0$ is arbitrarily small we have that series $\sum_{n\,=\,1}^{\infty} \frac{e(n)}{n^s}$ absolutely converges in a half-plane Re $s > 1$.

Let the function $E(s)$ define the Dirichlet series created by coefficients $e\,(n)$ in a half-plane Re $s > 1$

$$E\,(s) = \sum_{n\,=\,1}^{\infty} \frac{e\,(n)}{n^s}$$

And let $E(s)$ satisfy following assumptions

1. There exist positive constants A and β such that

$$E\,(s) = \frac{A}{(s-1)^{\beta}} + G(s)$$

   where $G(s)$ is a regular function in a half-plane Re $s > \frac{1}{2}$;

2. If $|t| \;\geq\; 3$, then there exists a constant A_0 such that

$$|E\,(1 \;+\; it)| \leq A_0 \log|t|.$$

If conditions 1 and 2 are met, then exists such a positive integer N that the following asymptotic formula

$$T\,(x) = \sum_{n\,\leq\,x} f\,(n) = \exp\Bigg(c_0\Big(\log x\Big)^{\frac{\beta}{\beta+1}}\Bigg\{\sum_{(h,v)} H\,(h,v)\,\Big(\log x\Big)^{-\frac{2h+v\beta}{2\beta+2}}$$

$$\times\Bigg(\Big(1 + c_0 \log x\Big)^{-\frac{1}{\beta+1}} - \frac{2h+v\beta}{2\beta}\Big(\log x\Big)^{-1}\Bigg) + O\Bigg(\Big(\log x\Big)^{-\frac{2N+4+\beta}{2\beta+2}}\Bigg)\Bigg\}\Bigg)$$

is true. Here c_0 is a countable constant that depends on A and β, N is an arbitrary fixed positive integer, $H\,(h,v)$ are suitable constants independent of x and N. The sum $\sum_{(h,v)}$ means summation over all pairs (h,v), $1 \;\leq\; h \;\leq\; N$, $\nu \;=\; 1,\; 2, \ldots$, that satisfy the inequality $h \;+ \frac{1}{2}\nu\beta \;\leq\; N \;+\; 2 \;+ \frac{1}{2}\beta$.

The function $I_n(z)$

$$I_n\,(z) = \frac{\exp z}{\sqrt{2\pi z}} \sum_{k\,=\,0}^{\infty} (-1)^k\, \frac{a_k\,(n)}{z^k},$$

where $\arg z < \frac{\pi}{2}$, $a_0\,(n) \;=\; 1$, $a_k\,(n) \;=\; \frac{\left(4n^2-1^2\right)\left(4n^2-3^2\right)\ldots\left(4n^2-(2k-1)^2\right)}{k!\,8^k}$, is the modified Bessel function.

The modified Bessel function $I_n(z)$ is one of two linearly independent solutions of the differential equation $x^2y'' + xy' - (x^2 + \alpha^2)y = 0$ written as the power series.

The function $I_n(z)$ is regular on $\mathbb{C}$ and goes to infinity for real positive z. Moreover, for $\alpha > 0$ $\lim_{x\to 0+} I_\alpha(x) = 0$ and for $\alpha = 0$ $\lim_{x\to 0+} I_0(x) = 1$.

Lemma 2. *For positive real numbers x, c, α the following relation*

$$I_\alpha(x) = \frac{x^\alpha}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{z+\frac{x^2}{4z}}}{z^{\alpha+1}} dz$$

holds and for sufficiently large x the following asymptotic formula

$$I_\alpha(x) = \frac{e^x}{\sqrt{2\pi x}}\left(1 - \frac{4\alpha^2 - 1}{8x} + O\left(\frac{1}{x^2}\right)\right)$$

is true.

Let us also notice that the constant in the O-term depends only on α.

3. Gaussian integer partition in a square-free numbers

Theorem 2. *By $\mathrm{g}_1(\alpha)$ we denote the number of square-free divisors of the Gaussian integer α. Then for $c_0 > 0, d_0 > 0$ and sufficiently large x the following asymptotic formula*

$$\sum_{N(\alpha)\le x} \mathrm{g}_1(\alpha) = c_0 x \sum_{n=0}^{\infty} d_n I_{n+1}\left(2\sqrt{\log x}\right)\left(\log x\right)^{-\frac{n+1}{2}} + O(x).$$

holds. Here coefficients $d_n, n \geq 1$, can be defined through the Taylor series coefficients of some function $\varphi_0(s)$ considered below.

Proof. Let $e_1(\alpha)$ be characteristic function over the set of square-free numbers. Then for the generating function of $\mathrm{g}_1(\alpha)$ following identity in a half-plane Re $s > 1$

$$F_1(s) = \sum_{0\,\neq\,\alpha\,\in\,G} \frac{\mathrm{g}_1(\alpha)}{N(\alpha)^s} = \prod_{N(\alpha)\,>\,1}\left(1 - \frac{e_1(\alpha)}{N(\alpha)^s}\right)^{-1}$$

is true.

$$\log F_1(s) = \sum_{\substack{f \text{ is square-free} \\ N(f) > 1}} \log\left(1 - \frac{1}{N(f)^s}\right) = \sum_{f\,\in\,F}\sum_{k\,=\,1}^{\infty} \frac{1}{k}\,\mu^2(f)\,N(f)^{-ks} - 1.$$

$$\log F_1(s) = \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{Z(ks)}{Z(2ks)} - 1 \right),$$

where $Z(s)$ is the well-known Hecke zetafunction with the Hecke character 1. Thus,

$$F_1(s) = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{Z(ks)}{Z(2ks)} - 1 \right) \right).$$

The series

$$\sum_{k=1}^{\infty} \frac{1}{k} \frac{Z(ks)}{Z(2ks)}$$

converges uniformly in each compact of half-plane $\mathrm{Re}\, s > 0$ except points $\frac{1}{k}$ and $\frac{\sigma+i\gamma}{2k}$, where $\sigma + i\gamma$ are complex roots of $Z(s)$. Thus, the function $F_1(s)$ is regular in a half-plane $\mathrm{Re}\, s > 0$ except specified points. Hence in a circle $|s-1| \leq \frac{1}{2}$ the following representation

$$F_1(s) = \exp\left(\frac{1}{Z(2)\,(s-1)} + \varphi_0(s) \right)$$

is true, where the function $\varphi_0(s)$ is regular in a circle $|s-1| \leq \frac{1}{2}$. Therefore, we can consider the Taylor series of $\varphi_0(s)$ in this circle

$$\varphi_0(s) = \sum_{n=0}^{\infty} \frac{\varphi_0^{(n)}(1)}{n!}(s-1)^n$$
$$= c_0 \exp\left(\frac{1}{Z(s)\,(s-1)} \right) \left(1 + a_1(s-1) + a_2(s-1)^2 + \dots \right),$$

where $c_0 = \exp\big(\varphi_0(1)\big) > 0$.

Now, using the well-known relation

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^{s+1}}{s(s+1)}\, ds = \begin{cases} x-1, & \text{if } x > 1, \\ 0, & \text{if } 0 < x \leq 1, \end{cases}$$

we get

$$\sum_{0 < N(\alpha) \leq x} \mathrm{g}_1(\alpha) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F_1(s) \frac{x^{s+1}}{s(s+1)}\, ds.$$

Function $F_1(s)$ doesn't have singularities in a half-plane Re $s \geq 1$ except the first kind pole $s = 1$. Let us replace the integration segment $(2-i\infty,\ 2+i\infty)$ with the union of following segments

Γ_1 denotes a segment $(\,1-i\infty, 1-ia\,]$;

Γ_2 denotes a half-circle with radius a and the center in a point $s=1$

$$1+a\exp(i\theta), -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}, (0<a<1);$$

Γ_3 denotes a segment $[\,1+ia, 1+i\infty\,)$.

It follows from lemma about estimates of Hecke function in critical strip that integration segments Γ_1 and Γ_3 can be estimated as $O\left(x^2\right)$. For the segment Γ_2 we use substitution of integration variable

$$s=1+\frac{1}{z}.$$

Hence $z=a^{-1}\exp(i\theta)$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, $ds=-\frac{1}{z^2}\,dz$.

After constricting the integration segment Γ_2 into a point the following equations

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\Gamma_2} F_1(s)\,\frac{x^{s+1}}{s(s+1)}\,ds &= \frac{1}{2\pi i}\int_{\Gamma_2} F_1(s)\,\frac{x^{s-1+2}}{s(s+1)}\,ds\\
&=\frac{x^2}{2\pi i}\int_{\Gamma_2} F_1(s)\,\frac{x^{s-1}}{s(s+1)}\,ds\\
&=\frac{x^2}{2\pi i}\int_{b-i\infty}^{b+i\infty}\exp\left(\varphi_0\left(1+\frac{1}{z}\right)\right)\frac{x^{\frac{1}{z}}}{\left(1+\frac{1}{z}\right)\left(2+\frac{1}{z}\right)}\,dz+O\left(x^2\right)\\
&=\frac{x^2}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{z^2x^{\frac{1}{z}}\exp\left(\varphi_0\left(1+\frac{1}{z}\right)\right)}{(z+1)(2z+1)}\,dz+O\left(x^2\right)\\
&=\frac{c_0x^2}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{\exp\left(z+\frac{1}{z}\log x\right)}{(z+1)(2z+1)}\left(1+\frac{b_1}{z}+\frac{b_2}{z^2}+\dots\right)dz+O\left(x^2\right)
\end{aligned}$$

are true for all $b>0$. Hence, we want to use the modified Bessel function $I_n(z)$.

$$\frac{1}{2\pi i}\int_{\Gamma_2} F_1(s)\,\frac{x^{\,s+1}}{s(s+1)}\,ds$$

$$\begin{aligned}
&=\frac{c_0x^2}{2}\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{\exp\left(z+\frac{1}{z}\log x\right)}{z^2}\left(1+\frac{b_1}{z}+\frac{b_2}{z^2}+\dots\right)dz\ +O\left(x^2\right)\\
&=c_0\frac{x^2}{2}\sum_{n=0}^{\infty}b_n\int_{b-i\infty}^{b+i\infty}\exp\left(z+\frac{1}{z}\log x\right)\,z^{-n-2}dz+O\left(x^2\right).
\end{aligned}$$

Note that

$$I_n(x) = \frac{x^n}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{\exp\left(z+\frac{x^2}{4z}\right)}{z^{n+1}}\,dz.$$

Therefore,

$$\frac{1}{2\pi i}\int_{\Gamma_2} F_1(s)\,\frac{x^{s+1}}{s(s+1)}\,ds = c_0\frac{x^2}{2}\sum_{n=0}^{\infty} b_n I_{n+1}\left(2\sqrt{\log x}\right) + O\left(x^2\right).$$

By the asymptotic differentiation we get the statement of the theorem.

4. Gaussian integer partition in a power-free numbers

Theorem 3. *By* $\mathrm{g}_2(\alpha)$ *we denote the number of power-free divisors of the Gaussian integer* α. *Then for sufficiently large* x *the following asymptotic formula*

$$\sum_{N(\alpha)\le x} \mathrm{g}_2(\alpha) = x\sum_{n=0}^{\infty} d_n \frac{I_{n+1}\left(2\sqrt{\log x}\right)}{(\log x)^{\frac{n+1}{2}}} + O(x)$$

holds. Here d_n, $n = 0,\ 1,\ \ldots,$ *can be defined through the Taylor series coefficients of some function* $F_{2,3}(s)$ *considered below.*

Let $e_2(\alpha)$ be characteristic functions over the set of power-free Gaussian integers. Then for the generating function of $\mathrm{g}_2(\alpha)$ following identity in a half-plane Re $s > 1$

$$F_2(s) = \sum_{0\ne\alpha\in G} \frac{\mathrm{g}_2(\alpha)}{N(\alpha)^s} = \prod_{N(\alpha)>1}\left(1-\frac{e_2(\alpha)}{N(\alpha)^s}\right)^{-1}$$

is true.

To find the generating series for $F_2(s)$ let us consider that the number $S(x)$ of power-free numbers with norms not more than x is equal to the number of all Gaussian integers in the circle of a radius $x^{\frac{1}{2}}$ with the center in the point $s = 0$ without the number of power-full numbers in this circle. (The Gaussian integer α is power-full if $\alpha = \mathfrak{p}_1^{k_1}\mathfrak{p}_2^{k_2}\ldots\mathfrak{p}_r^{k_r}$ and $GCD(k_1,\ k_2,\ \ldots,\ k_r) > 1$, where $k_i \in \mathbb{N}$, $i = \overline{1;r}$.) Thus,

$$S(x) = x - x^{\frac{1}{2}} - x^{\frac{1}{3}} + O\left(x^{\frac{1}{5}+\varepsilon}\right).$$

Hence, for Re $s > 1$ we have

$$F_2(x) = \sum_{\alpha \text{ is power-free}} \frac{1}{N(\alpha)^s} = Z(s) - Z(2s) - Z(3s) + G(s),$$

where the function $G(s)$ is regular in a half-plane Re $s > \frac{1}{5}$.

Therefore,

$$\log F_2(s) = \sum_{\delta \text{ is power-free}} \log\left(1 - \frac{1}{N(\delta)^s}\right) = \sum_{\delta \text{ is power-free}} \frac{1}{N(\delta)^s} + F_{2,1}(s),$$

where $F_{2,1}(s)$ is a regular function in half-plane Re $s > \frac{1}{2}$.

$$\log F_2(s) = Z(s) + F_{2,2}(s),$$

where $F_{2,2}(s)$ is a regular function in half-plane Re $s > \frac{1}{2}$.

$$F_2(s) = \exp\left(Z(s) + F_{2,2}(s)\right) = \exp\left(\frac{\pi}{s-1} + F_{2,3}(s)\right),$$

where $F_{2,3}(s)$ is a regular function in half-plane Re $s > \frac{1}{2}$.

Theorem 3 can be proved in a similar way to Theorem 2.

5. Gaussian integer partition in a power-free numbers norm ascending order

The function $g_2^*(\alpha)$ denotes the number of representation of Gaussian integer α in the power-free number product $\alpha = \delta_1\delta_2\ldots\delta_r$, δ_i are power-free, $i = \overline{1;r}$, and $N(\delta_1) \le N(\delta_2) \le \ldots \le N(\delta_r)$, where $N(\alpha)$ is the norm of α.

For Re $s > 1$ we have

$$\sum_{0 \ne \alpha \in G} \frac{g_2^*(\alpha)}{N(\alpha)^s} = \prod_{N(\alpha) \ge 2} \left(1 + \frac{e_2(\alpha)}{N(\alpha)^s}\right).$$

We will study function $g_2^*(\alpha)$ using the Kátai-Subbarao theorem.

We have

$$E(s) = \sum_{0 \ne \alpha \in G} \frac{e_2(\alpha)}{N(\alpha)^s} = \sum_{\alpha \text{ is power-free}} \frac{1}{N(\alpha)^s} = Z(s) + F_0(s),$$

where $e_2(\alpha)$ is a characteristic function over the set of power-free Gaussian integers, $F_0(s)$ is regular function in a half-plane Re $s > \frac{1}{2}$.

Moreover,

$$Z(s) = \sum_{0 \ne \alpha \in G} \frac{1}{N(\alpha)^s} = \sum_{n=0}^{\infty} \frac{r(n)}{n^s},$$

where $r(n)$ is the number of representation of n in a sum of two squares such that $r(n) = O(n^{\varepsilon})$. Here, the constant in the O-term depends on ε.

In this case all the conditions of the Kátai-Subbarao theorem are fulfilled.

Hence, we obtain the following theorem.

Theorem 4. *For sufficiently large x the following asymptotic formula*

$$\sum_{N(\alpha)\le x} g_2^*(\alpha) \sim \exp\left(c_0\sqrt{\log x}\right) \sum_{(h,\upsilon)} H(h,\upsilon)\left(\log x\right)^{-\frac{2h+\upsilon}{4}} \times \left(1 + a_0\left(\log x\right)^{-\frac{1}{2}} - \frac{2h+\upsilon}{4}\left(\log x\right)^{-1}\right)$$

holds, where c_0, a_0 are positive countable constants, mark $*$ above the sum $\sum_{(h,\vartheta)}$ means that we summarize by all the pairs (h,ϑ), $1 \le h \le N, \vartheta = 1, 2, \ldots$ such that

$$h + \frac{1}{2}\vartheta \le N + \frac{5}{2}.$$

Similar statements can be obtained for analogue for functions $g_2(\alpha)$ and $g_2^*(\alpha)$ that we will consider further.

6. Conclusion

Proposed research methods of Gaussian integers partition number can be applied to study of partition number function of integer ideals (divisors) from arbitrary imaginary quadratic field in a product of integer ideals (divisors) from this field.

Шрамко В. В.

Розбиття цілих гаусових чисел в добуток степенево-вільних

Резюме

Нехай функція $g_1(\alpha)$ являє собою число розкладань цілого гаусового числа α у вигляді добутку безквадратних чисел. Нехай функція $g_2(\alpha)$ являє собою число розкладань цілого гаусового числа α у вигляді добутку степенево-вільних чисел. В цій статті ми розглянемо суматорні функції $\sum_{N(\alpha)\le x} g_1(\alpha)$ та $\sum_{N(\alpha)\le x} g_2(\alpha)$ та отримаємо для них асимптотичні формули. Також, ми використаємо аналог теореми Kátai-Subbarao для вивчення розподілу значень функції $g_2(\alpha)$ у випадку, коли степенево-вільні множники розташовані в порядку зростання їх норм.

Ключові слова: Дзета-функція Гекке, безквадратне ціле гаусове число, степенево-вільне ціле гаусове число, твірний ряд Діріхле.

Шрамко В. В.

Разбиение целых гауссовых чисел в произведение степенно-свободных

Резюме

Пусть функция $g_1(\alpha)$ представляет собой число разбиения целого гауссового числа α в виде произведения безквадратных чисел. Пусть функция $g_2(\alpha)$ представляет собой число разбиения целого гауссового числа α в виде произведения степенно-свободных чисел. В этой статтье мы рассмотрим суматорные функции $\sum_{N(\alpha)\leq x} g_1(\alpha)$ и $\sum_{N(\alpha)\leq x} g_2(\alpha)$, а также получим ассимптотические формулы для них. Кроме того, мы воспользуемся аналогом теоремы Kátai-Subbarao для изучения распределения значений функции $g_2(\alpha)$ в случае, когда степенно-свободные множители располагаются в порядке возростания их норм.

Ключевые слова: Дзета-функция Гекке, безквадратное целое гауссово число, степенно-свободное целое гауссово число, производящий ряд Дирихле.

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