UDC 517.926

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ON THE REDUCTION OF THE LINEAR SYSTEM OF THE DIFFERENTIAL EQUATIONS WITH COEFFICIENTS OF OSCILLATING TYPE TO THE TRIANGULAR KIND IN THE RESONANT CASE

For the linear homogeneous differential system, whose coefficients are represented as an absolutely and uniformly convergent Fourier-series with slowly varying coefficients and frequency, the conditions of the existence of the transformation which leads it to triangular kind, are obtained in the resonant cases.

MSC 2010: 34A30, 34C25.

Key words: linear differential systems, triangular kind, Fourier-series, slowly varying parameters.

DOI: XXXX.

1. INTRODUCTION

In the theory of linear systems of differential equations is well known problem of the construction for the linear homogeneous system of the differential equations

$$\frac{dx}{dt} = A(t)x, \tag{1}$$

where $x = \mathrm{colon}(x_1, ..., x_n)$, $A(t) = (a_{jk}(t))_{j,k=\overline{1,n}}$, Lyapunov's transformation

$$x = L(t)y,$$

which leads the system (1) to the triangular kind

$$\frac{dy}{dt} = T(t)y,$$

where $T(t) = (b_{jk}(t))_{j,k=\overline{1,n}}$, $b_{jk}(t) \equiv 0$ $(j < k)$ [1–4].

In this paper, we assume, that the system (1) already reduced to a kind, close to triangular:

$$\frac{dx}{dt} = (T(t) + \mu P(t))x, \tag{2}$$

where μ – small parameter, and the matrix $P(t)$ has a some special kind. And we study the problem on bringing the system (2) to a purely triangular form

$$\frac{dy}{dt} = D(t)y,$$

where $D(t) = (d_{jk}(t))_{j,k=\overline{1,n}}$, $d_{jk} \equiv 0$ $(j < k)$.

This paper continues the research, begun in the paper [5]. The basic notation and definitions of the paper [5] are retained. As in the paper [5], we will study this problem for a third-order system $(n = 3)$ so as not to clutter up the presentation with secondary technical difficulties associated with the dimension of the system. All fundamental difficulties take place in this case too.

Statement of the Problem. We consider the next system of differential equations:

$$\frac{dx}{dt} = (B(t, \varepsilon) + \mu P(t, \varepsilon, \theta))x, \tag{3}$$

where $x = \mathrm{colon}(x_1, x_2, x_3)$,

$$B(t, \varepsilon) = \begin{pmatrix} b_{11}(t, \varepsilon) & 0 & 0 \\ b_{21}(t, \varepsilon) & b_{22}(t, \varepsilon) & 0 \\ b_{31}(t, \varepsilon) & b_{32}(t, \varepsilon) & b_{33}(t, \varepsilon) \end{pmatrix},$$

$b_{jk}(t, \varepsilon) \in S(m; \varepsilon_0)$ $(j, k = 1, 2, 3)$, $P(t, \varepsilon, \theta) = (p_{jk}(t, \varepsilon, \theta))_{j,k=1,2,3}$, $p_{jk}(t, \varepsilon, \theta) \in F(m; \varepsilon_0; \theta)$, $\mu \in (0, \mu_0) \subset \mathbb{R}^+$.

We assume that

$$b_{jj}(t, \varepsilon) - b_{kk}(t, \varepsilon) = im_{jk}\varphi(t, \varepsilon), \tag{4}$$

$m_{jk} \in \mathbb{Z}$, $m_{jj} \equiv 0$, $m_{jk} = -m_{kj}$ $(j, k = 1, 2, 3)$, $\varphi(t, \varepsilon)$ – the function that appears in the definition of the class $F(m; \varepsilon_0; \theta)$.

We study the problem of the existence of a transformation of kind

$$x = (E + \mu\Psi(t, \varepsilon, \theta, \mu))y, \tag{5}$$

$y = \mathrm{colon}(y_1, y_2, y_3)$, E – unit matrix of third order, Ψ – matrix of third order with elements from $F(l; \varepsilon_1; \theta)$ $(0 < l \leq m, \ 0 < \varepsilon_1 < \varepsilon_0)$, which leads at sufficiently small μ the system (3) to the kind:

$$\frac{dy}{dt} = K(t, \varepsilon, \theta, \mu))y, \tag{6}$$

where $K = (k_{jk}(t, \varepsilon, \theta, \mu))_{j,k=\overline{1,2,3}}$, $k_{jk} \equiv 0$ $(j < k)$, $k_{jk}(t, \varepsilon, \theta, \mu) \in F(l; \varepsilon_1; \theta)$.

That is, the same problem is considered as in work [5], but, taking into account conditions (4), in the resonance case, in contrast to [5].

2. AUXILIARY RESULTS

Lemma. *Let we have the system*

$$\frac{dv}{dt} = \left(A(t, \varepsilon) + \sum_{l=1}^{q} Q_l(t, \varepsilon, \theta) \mu^l \right) v, \tag{7}$$

$v = \text{colon}(v_1, v_2, v_3)$, $q \in \mathbb{N}$,

$$A(t, \varepsilon) = \begin{pmatrix} im_{12}\varphi(t, \varepsilon) & -c_{32}(t, \varepsilon) & 0 \\ 0 & im_{13}\varphi(t, \varepsilon) & 0 \\ 0 & c_{21}(t, \varepsilon) & im_{23}\varphi(t, \varepsilon) \end{pmatrix} \tag{8}$$

$m_{jk} \in \mathbb{N}$, $c_{jk}(t, \varepsilon) \in S(m; \varepsilon_0)$, *and* $\varphi(t, \varepsilon)$ – *the function in the definition of class* $F(m; \varepsilon_0; \theta)$, *the elements of matrices* Q_l $(l = \overline{1, q})$ *belongs to the class* $F(m; \varepsilon_0; \theta)$.

Then there exists $\mu_1 \in (0, \mu_0)$, *such that for all* $\mu \in (0, \mu_1)$ *there exists the Lyapunov's transformation of kind*

$$v = \left(E + \sum_{l=1}^{q} \Psi_l(t, \varepsilon, \theta) \mu^l \right) w, \tag{9}$$

where elemens of matrices $\Psi_l(t, \varepsilon, \theta)$ $(l = \overline{1, q})$ *belongs to the class* $F(m; \varepsilon_0; \theta)$, *which leads the system (7) to kind:*

$$\frac{dw}{dt} = \left(A(t, \varepsilon) + \sum_{l=1}^{q} U_l(t, \varepsilon) \mu^l + \varepsilon \sum_{l=1}^{q} V_l(t, \varepsilon, \theta) \mu^l + \mu^{q+1} W(t, \varepsilon, \theta, \mu) \right) w, \tag{10}$$

where $U_l(t, \varepsilon)$ – *the matrices with elements from* $S(m; \varepsilon_0)$, V_l, W – *the matrices with elements from* $F(m-1; \varepsilon_0; \theta)$.

Proof. We substitute the expression (8) into system (7), and require that the transformed system has the kind (9). We obtain the next chain of matrix differential equations for detemining matrices $\Psi_1, ..., \Psi_q$:

$$\frac{d\Psi_1}{dt} = A(t, \varepsilon)\Psi_1 - \Psi_1 A(t, \varepsilon) + Q_1(t, \varepsilon, \theta) - U_1(t, \varepsilon) - \varepsilon V_1(t, \varepsilon, \theta), \tag{11}$$

$$\frac{d\Psi_l}{dt} = A(t,\varepsilon)\Psi_l - \Psi_l A(t,\varepsilon) + Q_l(t,\varepsilon,\theta) - \sum_{\nu=1}^{l-1} Q_\nu \Psi_{l-\nu} -$$

$$-\sum_{\nu=1}^{l-1}\Psi_\nu U_{l-\nu}(t,\varepsilon) - \varepsilon\sum_{\nu=1}^{l-1}\Psi_\nu V_{l-\nu}(t,\varepsilon,\theta) - U_l(t,\varepsilon) - \varepsilon V_l(t,\varepsilon,\theta), \ l = \overline{2,q}. \ (12)$$

where $\Psi_l = (\psi_{jk}^l)_{j,k=1,2,3}$, $Q_l = (q_{jk}^l)_{j,k=1,2,3}$, $U_l = (u_{jk}^l)_{j,k=1,2,3}$, $V_l = (v_{jk}^l)_{j,k=1,2,3}$ $(l = \overline{1,q})$.

Then the matrix W at sufficiently small values μ is determined from the equation:

$$\left(E + \sum_{l=1}^{q}\Psi_l\mu^l\right)W = \sum_{s=0}^{q-1}\left[\sum_{\sigma+\delta=s+q+1}(Q_\sigma\Psi_\delta - \Psi_\sigma U_\delta)\right]\mu^s -$$

$$-\sum_{s=0}^{q-1}\left(\sum_{\sigma+\delta=s+q+1}\Psi_\sigma V_\delta\right)\mu^s. \qquad (13)$$

We consider the equation (11). In the component it looks like this:

$$\frac{d\psi_{11}^1}{dt} = -c_{32}(t,\varepsilon)\psi_{21}^1 + q_{11}^1(t,\varepsilon,\theta) - u_{11}^1(t,\varepsilon) - \varepsilon v_{11}^1(t,\varepsilon,\theta),$$

$$\frac{d\psi_{12}^1}{dt} = i(m_{12} - m_{13})\varphi(t,\varepsilon)\psi_{12}^1 - c_{32}(t,\varepsilon)(\psi_{22}^1 - \psi_{11}^1) - c_{21}(t,\varepsilon)\psi_{13}^1 +$$

$$+q_{12}^1(t,\varepsilon,\theta) - u_{12}^1(t,\varepsilon) - \varepsilon v_{12}^1(t,\varepsilon,\theta),$$

$$\frac{d\psi_{13}^1}{dt} = i(m_{12} - m_{23})\varphi(t,\varepsilon)\psi_{13}^1 - c_{32}(t,\varepsilon)\psi_{23}^1 +$$

$$+q_{13}^1(t,\varepsilon,\theta) - u_{13}^1(t,\varepsilon) - \varepsilon v_{13}^1(t,\varepsilon,\theta),$$

$$\frac{d\psi_{21}^1}{dt} = i(m_{13} - m_{12})\varphi(t,\varepsilon)\psi_{21}^1 + q_{21}^1(t,\varepsilon,\theta) - u_{21}^1(t,\varepsilon) - \varepsilon v_{21}^1(t,\varepsilon,\theta),$$

$$\frac{d\psi_{22}^1}{dt} = c_{32}(t,\varepsilon)\psi_{21}^1 - c_{21}(t,\varepsilon)\psi_{32}^1 + q_{22}^1(t,\varepsilon,\theta) - u_{11}^1(t,\varepsilon) - \varepsilon v_{22}^1(t,\varepsilon,\theta),$$

$$\frac{d\psi_{23}^1}{dt} = i(m_{13} - m_{23})\varphi(t,\varepsilon)\psi_{23}^1 + q_{23}^1(t,\varepsilon,\theta) - u_{23}^1(t,\varepsilon) - \varepsilon v_{23}^1(t,\varepsilon,\theta),$$

$$\frac{d\psi_{31}^1}{dt} = i(m_{23} - m_{12})\varphi(t,\varepsilon)\psi_{31}^1 + c_{21}(t,\varepsilon)\psi_{21}^1 +$$

$$+q_{31}^1(t,\varepsilon,\theta) - u_{31}^1(t,\varepsilon) - \varepsilon v_{31}^1(t,\varepsilon,\theta),$$

$$\frac{d\psi_{32}^1}{dt} = i(m_{23} - m_{13})\varphi(t,\varepsilon)\psi_{32}^1 + c_{21}(t,\varepsilon)(\psi_{21}^1 - \psi_{33}^1) + c_{32}(t,\varepsilon)\psi_{31}^1 +$$

$$+q_{32}^1(t,\varepsilon,\theta) - u_{32}^1(t,\varepsilon) - \varepsilon v_{32}^1(t,\varepsilon,\theta),$$

$$\frac{d\psi_{33}^1}{dt} = c_{21}(t,\varepsilon)\psi_{23}^1 + q_{33}^1(t,\varepsilon,\theta) - u_{33}^1(t,\varepsilon) - \varepsilon v_{33}^1(t,\varepsilon,\theta).$$

$$(14)$$

Define ψ_{jk}^1, u_{jk}^1, v_{jk}^1 by the following expression:

$$\psi_{21}^1(t,\varepsilon,\theta) = - \sum_{\substack{n=-\infty \\ (n \neq m_{13}-m_{12})}}^{\infty} \frac{\Gamma_n[q_{21}^1(t,\varepsilon,\theta)]}{i(m_{13} - m_{12} - n)\varphi(t,\varepsilon)} e^{in\theta(t,\varepsilon)},$$

$$u_{21}^1(t,\varepsilon) = \Gamma_{m_{13}-m_{12}}[q_{21}^1(t/\varepsilon,\theta)],$$

$$v_{21}^1(t,\varepsilon,\theta) = \frac{1}{\varepsilon} \sum_{\substack{n=-\infty \\ (n \neq m_{13}-m_{12})}}^{\infty} \frac{d}{dt} \left(\frac{\Gamma_n[q_{21}^1(t,\varepsilon,\theta)]}{i(m_{13}-m_{12}-n)\varphi(t,\varepsilon)} \right) e^{in\theta(t,\varepsilon)},$$

$$\psi_{11}^1(t,\varepsilon,\theta) = \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \frac{\Gamma_n[q_{11}^1(t,\varepsilon,\theta) - c_{32}(t,\varepsilon)\psi_{21}^1(t,\varepsilon,\theta)]}{in\varphi(t,\varepsilon)} e^{in\theta(t,\varepsilon)},$$

$$u_{11}^1(t,\varepsilon) = \Gamma_0[q_{11}^1(t,\varepsilon,\theta) - c_{32}(t,\varepsilon)\psi_{21}^1(t,\varepsilon,\theta)],$$

$$v_{11}^1(t,\varepsilon,\theta) = -\frac{1}{\varepsilon} \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \frac{d}{dt} \left(\frac{\Gamma_n[q_{11}^1(t,\varepsilon,\theta) - c_{32}(t,\varepsilon)\psi_{21}^1(t,\varepsilon,\theta)]}{in\varphi(t,\varepsilon)} \right) e^{in\theta(t,\varepsilon)},$$

$$\psi_{31}^1(t,\varepsilon,\theta) = - \sum_{\substack{n=-\infty \\ (n \neq m_{23}-m_{12})}}^{\infty} \frac{\Gamma_n[q_{31}^1(t,\varepsilon,\theta) + c_{21}(t,\varepsilon)\psi_{21}^1(t,\varepsilon,\theta)]}{i(m_{23}-m_{12}-n)\varphi(t,\varepsilon)} e^{in\theta(t,\varepsilon)},$$

$$u_{31}^1(t,\varepsilon) = \Gamma_{m_{23}-m_{12}}[q_{11}^1(t,\varepsilon,\theta) - c_{32}(t,\varepsilon)\psi_{21}^1(t,\varepsilon,\theta)],$$

$$v_{31}^1(t,\varepsilon,\theta) = \frac{1}{\varepsilon} \sum_{\substack{n=-\infty \\ (n \neq m_{23}-m_{12})}}^{\infty} \frac{d}{dt} \left(\frac{\Gamma_n[q_{31}^1(t,\varepsilon,\theta) + c_{21}(t,\varepsilon)\psi_{21}^1(t,\varepsilon,\theta)]}{i(m_{23}-m_{12}-n)\varphi(t,\varepsilon)} \right) e^{in\theta(t,\varepsilon)},$$

$$\psi_{23}^1(t,\varepsilon,\theta) = - \sum_{\substack{n=-\infty \\ (n \neq m_{13}-m_{23})}}^{\infty} \frac{\Gamma_n[q_{23}^1(t,\varepsilon,\theta)]}{i(m_{13}-m_{23}-n)\varphi(t,\varepsilon)} e^{in\theta(t,\varepsilon)},$$

$$u_{23}^1(t,\varepsilon) = \Gamma_{m_{13}-m_{23}}[q_{23}^1(t,\varepsilon,\theta)],$$

$$v_{23}^1(t,\varepsilon,\theta) = \frac{1}{\varepsilon} \sum_{\substack{n=-\infty \\ (n \neq m_{13}-m_{23})}}^{\infty} \frac{d}{dt} \left(\frac{\Gamma_n[q_{23}^1(t,\varepsilon,\theta)]}{i(m_{13}-m_{23}-n)\varphi(t,\varepsilon)} \right) e^{in\theta(t,\varepsilon)},$$

$$\psi_{33}^1(t,\varepsilon,\theta) = \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \frac{\Gamma_n[q_{33}^1(t,\varepsilon,\theta) + c_{21}(t,\varepsilon)\psi_{23}^1(t,\varepsilon,\theta)]}{in\varphi(t,\varepsilon)} e^{in\theta(t,\varepsilon)},$$

$$u_{33}^1(t,\varepsilon) = \Gamma_0[q_{33}^1(t,\varepsilon,\theta) + c_{21}(t,\varepsilon)\psi_{23}^1(t,\varepsilon,\theta)],$$

$$v_{33}^1(t,\varepsilon,\theta) = -\frac{1}{\varepsilon} \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \frac{d}{dt} \left(\frac{\Gamma_n[q_{33}^1(t,\varepsilon,\theta) + c_{21}(t,\varepsilon)\psi_{23}^1(t,\varepsilon,\theta)]}{in\varphi(t,\varepsilon)} \right) e^{in\theta(t,\varepsilon)},$$

$$\psi_{22}^1(t,\varepsilon,\theta) = \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \frac{\Gamma_n[q_{22}^1(t,\varepsilon,\theta) + c_{32}(t,\varepsilon)\psi_{21}^1(t,\varepsilon,\theta) - c_{21}(t,\varepsilon)\psi_{23}^1(t,\varepsilon,\theta)]}{in\varphi(t,\varepsilon)} e^{in\theta(t,\varepsilon)},$$

$$u_{22}^1(t,\varepsilon) = \Gamma_0[q_{22}^1(t,\varepsilon,\theta) + c_{32}(t,\varepsilon)\psi_{21}^1(t,\varepsilon,\theta) - c_{21}(t,\varepsilon)\psi_{23}^1(t,\varepsilon,\theta)],$$

$$v_{22}^1(t,\varepsilon,\theta) = -\frac{1}{\varepsilon}\sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty}\frac{d}{dt}\left(\frac{\Gamma_n[q_{22}^1(t,\varepsilon,\theta)+c_{32}(t,\varepsilon)\psi_{21}^1(t,\varepsilon,\theta)-c_{21}(t,\varepsilon)\psi_{23}^1(t,\varepsilon,\theta)]}{in\varphi(t,\varepsilon)}\right)e^{in\theta(t,\varepsilon)},$$

$$\psi_{32}^1(t,\varepsilon,\theta) = -\sum_{\substack{n=-\infty \\ (n\neq m_{23}-m_{13})}}^{\infty}\frac{\Gamma_n[q_{32}^1(t,\varepsilon,\theta)+c_{21}(t,\varepsilon)(\psi_{22}^1-\psi_{33}^1)+c_{32}(t,\varepsilon)\psi_{31}^1]}{i(m_{23}-m_{13}-n)\varphi(t,\varepsilon)}e^{in\theta(t,\varepsilon)},$$

$$u_{32}^1(t,\varepsilon) = \Gamma_{m_{23}-m_{13}}[q_{32}^1(t,\varepsilon,\theta)+c_{21}(t,\varepsilon)(\psi_{22}^1-\psi_{33}^1)+c_{32}(t,\varepsilon)\psi_{31}^1],$$

$$v_{32}^1(t,\varepsilon,\theta) = \frac{1}{\varepsilon}\sum_{\substack{n=-\infty \\ (n\neq m_{23}-m_{13})}}^{\infty}\frac{d}{dt}\left(\frac{\Gamma_n[q_{32}^1(t,\varepsilon,\theta)+c_{21}(t,\varepsilon)(\psi_{22}^1-\psi_{33}^1)+c_{32}(t,\varepsilon)\psi_{31}^1]}{i(m_{23}-m_{13}-n)\varphi(t,\varepsilon)}\right)e^{in\theta(t,\varepsilon)},$$

$$\psi_{13}^1(t,\varepsilon,\theta) = -\sum_{\substack{n=-\infty \\ (n\neq m_{12}-m_{23})}}^{\infty}\frac{\Gamma_n[q_{13}^1(t,\varepsilon,\theta)-c_{32}(t,\varepsilon)\psi_{23}^1(t,\varepsilon,\theta)]}{i(m_{12}-m_{23}-n)\varphi(t,\varepsilon)}e^{in\theta(t,\varepsilon)},$$

$$u_{13}^1(t,\varepsilon) = \Gamma_{m_{12}-m_{23}}[q_{13}^1(t,\varepsilon,\theta)-c_{32}(t,\varepsilon)\psi_{23}^1(t,\varepsilon,\theta)],$$

$$v_{13}^1(t,\varepsilon,\theta) = \frac{1}{\varepsilon}\sum_{\substack{n=-\infty \\ (n\neq m_{12}-m_{23})}}^{\infty}\frac{d}{dt}\left(\frac{\Gamma_n[q_{13}^1(t,\varepsilon,\theta)-c_{32}(t,\varepsilon)\psi_{23}^1(t,\varepsilon,\theta)]}{i(m_{12}-m_{23}-n)\varphi(t,\varepsilon)}\right)e^{in\theta(t,\varepsilon)},$$

$$\psi_{12}^1(t,\varepsilon,\theta) = -\sum_{\substack{n=-\infty \\ (n\neq m_{12}-m_{13})}}^{\infty}\frac{\Gamma_n[q_{12}^1(t,\varepsilon,\theta)-c_{32}(t,\varepsilon)(\psi_{22}^1-\psi_{11}^1)-c_{21}(t,\varepsilon)\psi_{13}^1]}{i(m_{12}-m_{13}-n)\varphi(t,\varepsilon)}e^{in\theta(t,\varepsilon)},$$

$$u_{12}^1(t,\varepsilon) = \Gamma_{m_{12}-m_{13}}[q_{12}^1(t,\varepsilon,\theta)-c_{32}(t,\varepsilon)(\psi_{22}^1-\psi_{11}^1)-c_{21}(t,\varepsilon)\psi_{13}^1],$$

$$v_{12}^1(t,\varepsilon,\theta) = \frac{1}{\varepsilon}\sum_{\substack{n=-\infty \\ (n\neq m_{12}-m_{13})}}^{\infty}\frac{d}{dt}\left(\frac{\Gamma_n[q_{12}^1(t,\varepsilon,\theta)-c_{32}(t,\varepsilon)(\psi_{22}^1-\psi_{11}^1)-c_{21}(t,\varepsilon)\psi_{13}^1]}{i(m_{12}-m_{13}-n)\varphi(t,\varepsilon)}\right)e^{in\theta(t,\varepsilon)}.$$

All the elements of matrix U_1 belongs to the class $S(m;\varepsilon_0)$. All the elements of matrix Ψ_1 belongs to the class $F(m;\varepsilon_0;\theta)$. All the elements of matrix V_1 belongs to the class $F(m-1;\varepsilon_0;\theta)$.

All the equations (12) are considered similarly to equations (11), and so the matrices Ψ_l, U_l, V_l $(l = \overline{1,q})$ are determined. And also all the elements of matrix Ψ_l belongs to the class $F(m;\varepsilon_0;\theta)$, all the elements of matrix U_l belongs to the class $S(m;\varepsilon_0)$, all the elements of matrix V_l belongs to the class $F(m-1;\varepsilon_0;\theta)$ $(l = \overline{1,q})$. Matrix W are determined from the equations (13).

Lemma are proved.

3. Problem solving method and basic results.

We seek the transformation of the kind:

$$x = (E + \mu \Psi(t, \varepsilon, \theta, \mu)) y, \tag{15}$$

$y = \mathrm{colon}(y_1, y_2, y_3)$, E – unit matrix of third order,

$$\Psi(t, \varepsilon, \theta, \mu) = \begin{pmatrix} 0 & \psi_{12}(t, \varepsilon, \theta, \mu) & \psi_{13}(t, \varepsilon, \theta, \mu) \\ 0 & 0 & \psi_{23}(t, \varepsilon, \theta, \mu) \\ 0 & 0 & 0 \end{pmatrix},$$

$\psi_{jk} \in F(m_1; \varepsilon_1; \theta)$ $(0 \le m_1 \le m; \ 0 \le \varepsilon_1 < \varepsilon_0)$, which leads the system (3) to the kind:

$$\frac{dy}{dt} = (B(t, \varepsilon) + \mu D(t, \varepsilon, \theta, \mu)) y, \tag{16}$$

where

$$D(t, \varepsilon, \theta, \mu) = \begin{pmatrix} d_{11}(t, \varepsilon, \theta, \mu) & 0 & 0 \\ d_{21}(t, \varepsilon, \theta, \mu) & d_{22}(t, \varepsilon, \theta, \mu) & 0 \\ d_{31}(t, \varepsilon, \theta, \mu) & d_{32}(t, \varepsilon, \theta, \mu) & d_{33}(t, \varepsilon, \theta, \mu) \end{pmatrix}.$$

We substitute the expression (15) into system (3), and require that the transformed system has the kind (16). We obtain the next system of the differential equations for detemining $\psi_{12}, \psi_{13}, \psi_{23}$:

$$\frac{d\psi_{12}}{dt} = K_{12}(t, \varepsilon, \theta, \psi_{12}, \psi_{13}, \psi_{23}, \mu),$$

$$\frac{d\psi_{13}}{dt} = K_{13}(t, \varepsilon, \theta, \psi_{12}, \psi_{13}, \psi_{23}, \mu), \tag{17}$$

$$\frac{d\psi_{23}}{dt} = K_{23}(t, \varepsilon, \theta, \psi_{12}, \psi_{13}, \psi_{23}, \mu),$$

where

$$K_{12} = (m_{11} - m_{22})\varphi(t, \varepsilon)\psi_{12} - b_{32}(t, \varepsilon)\psi_{13} + p_{12}(t, \varepsilon, \theta) +$$

$$+ \mu b_{21}(t, \varepsilon)\psi_{12}^2 + \mu b_{32}(t, \varepsilon)\psi_{12}\psi_{23} -$$

$$- \mu^2 p_{21}(t, \varepsilon, \theta)\psi_{12}^2 + \mu^2 b_{31}(t, \varepsilon)\psi_{12}^2\psi_{23} + \mu^2 p_{32}(t, \varepsilon, \theta)\psi_{12}\psi_{23} +$$

$$+ \mu^2 b_{31}(t, \varepsilon)\psi_{12}\psi_{13} + \mu^2 p_{32}(t, \varepsilon, \theta)\psi_{13} + \mu^3 p_{31}(t, \varepsilon, \theta)\psi_{12}^2\psi_{23} + \mu^3 p_{31}(t, \varepsilon, \theta)\psi_{12}\psi_{13},$$

$$K_{13} = (m_{11} - m_{33})\varphi(t,\varepsilon)\psi_{13} + p_{13}(t,\varepsilon,\theta) + \mu(p_{11}(t,\varepsilon,\theta) - p_{33}(t,\varepsilon,\theta))\psi_{13}+$$

$$+\mu p_{12}(t,\varepsilon,\theta)\psi_{23} - \mu b_{13}(t,\varepsilon)\psi_{13}^2 - \mu b_{32}(t,\varepsilon)\psi_{13}\psi_{23}-$$

$$-\mu^2 p_{31}(t,\varepsilon,\theta)\psi_{13}^2 - \mu^2 p_{32}(t,\varepsilon,\theta)\psi_{13}\psi_{23},$$

$$K_{23} = (m_{22} - m_{33})\varphi(t,\varepsilon)\psi_{23} + b_{21}(t,\varepsilon)\psi_{13} + p_{23}(t,\varepsilon,\theta) + \mu p_{21}(t,\varepsilon,\theta)\psi_{12}+$$

$$+\mu(p_{22}(t,\varepsilon,\theta) - p_{33}(t,\varepsilon,\theta))\psi_{23} - \mu b_{31}(t,\varepsilon)\psi_{13}\psi_{23}-$$

$$-\mu b_{32}(t,\varepsilon)\psi_{23}^2 - \mu^2 p_{31}(t,\varepsilon,\theta)\psi_{13}\psi_{23} - \mu^2 p_{32}(t,\varepsilon,\theta)\psi_{23}^2.$$

In this case $d_{jk}(t,\varepsilon,\theta,\mu)$ $(j \geq k)$ has a kind:

$$d_{31}(t,\varepsilon,\theta) = p_{31}(t,\varepsilon,\theta),$$

$$d_{32}(t,\varepsilon,\theta,\mu) = p_{32}(t,\varepsilon,\theta) + b_{31}(t,\varepsilon)\psi_{12} + \mu p_{31}(t,\varepsilon,\theta)\psi_{12},$$

$$d_{33}(t,\varepsilon,\theta,\mu) = p_{33}(t,\varepsilon,\theta) + b_{31}(t,\varepsilon)\psi_{13} + b_{32}(t,\varepsilon)\psi_{23} + \mu(p_{31}(t,\varepsilon,\theta)\psi_{13} + p_{32}(t,\varepsilon,\theta)\psi_{23}),$$

$$d_{21}(t,\varepsilon,\theta,\mu) = p_{21}(t,\varepsilon,\theta) - b_{31}(t,\varepsilon)\psi_{13} - \mu p_{31}(t,\varepsilon,\theta)\psi_{23},$$

$$d_{22}(t,\varepsilon,\theta,\mu) = p_{22}(t,\varepsilon,\theta) - b_{21}(t,\varepsilon)\psi_{12} - b_{32}(t,\varepsilon)\psi_{23} + \mu p_{21}(t,\varepsilon,\theta)\psi_{12} - \mu d_{32}(t,\varepsilon,\theta,\mu)\psi_{23},$$

$$d_{11}(t,\varepsilon,\theta,\mu) = p_{11}(t,\varepsilon,\theta) - b_{21}(t,\varepsilon)\psi_{12} - b_{31}(t,\varepsilon)\psi_{13} - \mu(d_{21}(t,\varepsilon,\theta,\mu)\psi_{12} + p_{31}(t,\varepsilon,\theta)\psi_{13}).$$

$$(18)$$

Together with the system (17) we consider the auxiliary system:

$$\varphi(t,\varepsilon)\frac{d\xi_{12}}{d\theta} = K_{12}(t,\varepsilon,\theta,\xi_{12},\xi_{13},\xi_{23},\mu),$$

$$\varphi(t,\varepsilon)\frac{d\xi_{13}}{d\theta} = K_{13}(t,\varepsilon,\theta,\xi_{12},\xi_{13},\xi_{23},\mu), \qquad (19)$$

$$\varphi(t,\varepsilon)\frac{d\xi_{23}}{d\theta} = K_{23}(t,\varepsilon,\theta,\xi_{12},\xi_{13},\xi_{23},\mu),$$

where $\varphi(t,\varepsilon)$ – function in the definition of the class $F(m;\varepsilon_0;\theta)$, and t,ε are considered as constant. Using the method of the small parameter of Poincarais [6], we construct the partial sums of the series in degrees of the small parameter representing the 2π-periodic with respect to θ solution of the system (19):

$$\xi_{jk}^*(t,\varepsilon,\theta,\mu) = \xi_{jk}^0(t,\varepsilon,\theta) + \mu\xi_{jk}^1(t,\varepsilon,\theta) + \ldots + \mu^{2q-1}\xi_{jk}^{2q-1}(t,\varepsilon,\theta), \qquad (20)$$

where $\xi_{jk}^s(t,\varepsilon,\theta)$ $(s = \overline{0, 2q-1})$ – 2π-periodic with respect to θ functions. Regarding these functions, we obtain the chain of the system of the differential equations:

$$\frac{d\xi_{12}^0}{d\theta} = im_{12}\xi_{12}^0 - \frac{b_{32}(t,\varepsilon)}{\varphi(t,\varepsilon)}\,\xi_{13}^0 + \frac{p_{12}(t,\varepsilon,\theta)}{\varphi(t,\varepsilon)},$$

$$\frac{d\xi_{13}^0}{d\theta} = im_{13}\xi_{13}^0 + \frac{p_{13}(t,\varepsilon,\theta)}{\varphi(t,\varepsilon)}, \tag{21}$$

$$\frac{d\xi_{23}^0}{d\theta} = im_{23}\xi_{23}^0 + \frac{b_{21}(t,\varepsilon)}{\varphi(t,\varepsilon)}\,\xi_{13}^0 + \frac{p_{23}(t,\varepsilon,\theta)}{\varphi(t,\varepsilon)},$$

$$\frac{d\xi_{12}^1}{d\theta} = im_{12}\xi_{12}^1 - \frac{b_{32}(t,\varepsilon)}{\varphi(t,\varepsilon)}\,\xi_{13}^1 + \frac{1}{\varphi(t,\varepsilon)}F_{12}(t,\varepsilon,\theta,\xi_{12}^0,\xi_{13}^0,\xi_{23}^0),$$

$$\frac{d\xi_{13}^1}{d\theta} = im_{13}\xi_{13}^1 + \frac{1}{\varphi(t,\varepsilon)}F_{13}(t,\varepsilon,\theta,\xi_{12}^0,\xi_{13}^0,\xi_{23}^0), \tag{22}$$

$$\frac{d\xi_{23}^1}{d\theta} = im_{23}\xi_{23}^1 + \frac{b_{21}(t,\varepsilon)}{\varphi(t,\varepsilon)}\,\xi_{13}^1 + \frac{1}{\varphi(t,\varepsilon)}F_{23}(t,\varepsilon,\theta,\xi_{12}^0,\xi_{13}^0,\xi_{23}^0),$$

where

$$F_{12}(t,\varepsilon,\theta,\xi_{12}^0,\xi_{13}^0,\xi_{23}^0) = b_{21}(t,\varepsilon)(\xi_{12}^0)^2 + b_{32}(t,\varepsilon)\xi_{12}^0\xi_{23}^0,$$

$$F_{13}(t,\varepsilon,\theta,\xi_{12}^0,\xi_{13}^0,\xi_{23}^0) =$$

$$= (p_{11}(t,\varepsilon,\theta) - p_{33}(t,\varepsilon,\theta))\xi_{13}^0 + p_{12}(t,\varepsilon,\theta)\xi_{23}^0 - b_{31}(t,\varepsilon)(\xi_{13}^0)^2 - b_{32}(t,\varepsilon)\xi_{13}^0\xi_{23}^0,$$

$$F_{23}(t,\varepsilon,\theta,\xi_{12}^0,\xi_{13}^0,\xi_{23}^0) =$$

$$= (p_{22}(t,\varepsilon,\theta) - p_{33}(t,\varepsilon,\theta))\xi_{23}^0 + p_{21}(t,\varepsilon,\theta)\xi_{12}^0 - b_{32}(t,\varepsilon)(\xi_{23}^0)^2 - b_{31}(t,\varepsilon)\xi_{13}^0\xi_{23}^0,$$

$$\frac{d\xi_{12}^s}{d\theta} = im_{12}\xi_{12}^s - \frac{b_{32}(t,\varepsilon)}{\varphi(t,\varepsilon)}\,\xi_{13}^s +$$

$$+ \frac{1}{\varphi(t,\varepsilon)}\,P_{12}^s(t,\varepsilon,\theta,\xi_{12}^0,\xi_{13}^0,\xi_{23}^0,\ldots,\xi_{12}^{s-1},\xi_{13}^{s-1},\xi_{23}^{s-1}),$$

$$\frac{d\xi_{13}^s}{d\theta} = im_{13}\xi_{13}^0 +$$

$$+ \frac{1}{\varphi(t,\varepsilon)}\,Q_{13}^s(t,\varepsilon,\theta,\xi_{12}^0,\xi_{13}^0,\xi_{23}^0,\ldots,\xi_{12}^{s-1},\xi_{13}^{s-1},\xi_{23}^{s-1}),$$

$$\frac{d\xi_{23}^s}{d\theta} = im_{23}\xi_{23}^s + \frac{b_{21}(t,\varepsilon)}{\varphi(t,\varepsilon)}\,\xi_{13}^s +$$

$$+ \frac{1}{\varphi(t,\varepsilon)}\,R_{23}^s(t,\varepsilon,\theta,\xi_{12}^0,\xi_{13}^0,\xi_{23}^0,\ldots,\xi_{12}^{s-1},\xi_{13}^{s-1},\xi_{23}^{s-1}),\ \ s = 2,3,\ldots,2q-1. \tag{23}$$

$P_{12}^s, Q_{13}^s, R_{23}^s$ – polynomials from $\xi_{12}^0, \ldots, \xi_{23}^{s-1}$ with coefficients from the class $F(m; \varepsilon_0; \theta)$.

Theorem. *Let the system (17) such that:*

1) there exists the such functions $\xi_{jk}^*(t, \varepsilon, \theta, \mu)$, $\Phi_{jkl}(t, \varepsilon, \theta, \mu)$ ($j, k = 1, 2, 3; j < k; l = 1, 2, 3$) belongs to the class $F(m; \varepsilon_0; \theta)$, that the transformation

$$\psi_{jk} = \xi_{jk}^*(t, \varepsilon, \theta, \mu) + \Phi_{jk1}(t, \varepsilon, \theta, \mu)\sigma_{12} + \Phi_{jk2}(t, \varepsilon, \theta, \mu)\sigma_{13} + \Phi_{jk3}(t, \varepsilon, \theta, \mu)\sigma_{23}$$
$$(24)$$

$j, k = 1, 2, 3; j < k,$

leads the system (17) to a kind:

$$\frac{d\sigma}{dt} = \left(A_1(t, \varepsilon) + \sum_{l=1}^{q} U_l(t, \varepsilon)\mu^l \right) \sigma + \varepsilon g(t, \varepsilon, \theta, \mu) + \mu^{2q} c(t, \varepsilon, \theta, \mu) +$$

$$+ \varepsilon \left(\sum_{l=1}^{q} V_l(t, \varepsilon, \theta)\mu^l \right) \sigma + \mu^{q+1} L(t, \varepsilon, \theta, \mu) + \mu H(t, \varepsilon, \theta, \sigma, \mu), \qquad (25)$$

where $\sigma = \mathrm{colon}(\sigma_{12}, \sigma_{13}, \sigma_{23})$,

$$A_1(t, \varepsilon) = \begin{pmatrix} im_{12}\varphi(t, \varepsilon) & -b_{32}t, \varepsilon) & 0 \\ 0 & im_{13}\varphi(t, \varepsilon) & 0 \\ 0 & b_{21}(t, \varepsilon) & im_{23}\varphi(t, \varepsilon) \end{pmatrix},$$

$U_l(t, \varepsilon)$ ($l = \overline{1, q}$) – matrices with elements from $S(m; \varepsilon_0)$, $g = \mathrm{colon}(g_1, g_2, g_3)$, $g_j(t, \varepsilon, \theta, \mu) \in F(m - 1; \varepsilon_0; \theta)$ ($j = 1, 2, 3$), $c = \mathrm{colon}(c_1, c_2, c_3)$, $c_j(t, \varepsilon, \theta, \mu) \in F(m; \varepsilon_0; \theta)$, $V_l(t, \varepsilon, \theta)$ ($l = \overline{1, q}$) – matrices with elements from $F(m - 1; \varepsilon_0; \theta)$, $L(t, \varepsilon, \theta, \mu)$ – matrix with elements from $F(m; \varepsilon_0; \theta)$, the components of vector-function H – polynomials in respect to $\sigma_{12}, \sigma_{13}, \sigma_{23}$, containing terms not lower than second order with respect theese variables, with coefficients, belongs to the class $F(m; \varepsilon_0; \theta)$;

2) the eigenvalues $\lambda_j(t, \varepsilon, \mu)$ ($j = 1, 2, 3$) of the matrix

$$U(t, \varepsilon, \mu) = A_1(t, \varepsilon) + \sum_{l=1}^{q} U_l(t, \varepsilon)\mu^l$$

such that

$$\inf_{G(\varepsilon_0)} |\mathrm{Re}\lambda_j(t, \varepsilon, \theta)| \geq \gamma_0 \mu^{q_0} \ (\gamma_0 \geq 0, \ 0 < q_0 \leq q);$$

3) *for the matrix $U(t, \varepsilon, \mu)$ there exists the matrix $Y(t, \varepsilon, \mu)$ such that*

a) $\inf\limits_{G(\varepsilon_0)} |\det Y(t, \varepsilon, \mu)| > 0$,

b) $Y^{-1}UY = \Lambda(t, \varepsilon, \mu)$ – *diagonal matrix.*

Then there exists $\mu_1 \in (0, \mu_0)$, $\varepsilon_1(\mu) \in (0, \mu_0)$ such that for all $\mu \in (0, \mu_1)$ and for all $\varepsilon \in (0, \varepsilon_1(\mu))$ there exists the transformation of the kind (15), whose coefficients $\psi_{jk}(t, \varepsilon, \theta, \mu)$ $(j < k)$ belongs to the class $F(m-1; \varepsilon_2(\mu); \theta)$, which leads the system (3) to a triangular kind (16), where $d_{jk}(t, \varepsilon, \theta, \mu)$ $(j \geq k)$ are determines by the formulas (18).

Proof is complete analogous to the proof of the Theorem 3 from [5].

Now for different relationships between m_{jk} we get for the system (17) more specific conditions of existence of the transformation (24). We will check only condition 1) of the theorem, assuming conditions 2) and 3) to be satisfied.

Case 1. $m_{12} \neq m_{13}$, $m_{13} \neq m_{23}$, $m_{12} \neq m_{23}$.

Consider a generating system (21). Under the conditions

$$\int\limits_0^{2\pi} p_{13}(t, \varepsilon, \theta) \, e^{-im_{13}\theta} d\theta = 0,$$

$$\int\limits_0^{2\pi} (p_{12}(t, \varepsilon, \theta) - b_{32}(t, \varepsilon)\chi_{13}(t, \varepsilon, \theta)) \, e^{-im_{12}\theta} d\theta = 0, \qquad (26)$$

$$\int\limits_0^{2\pi} (p_{12}(t, \varepsilon, \theta) + b_{21}(t, \varepsilon)\chi_{13}(t, \varepsilon, \theta)) \, e^{-im_{23}\theta} d\theta = 0,$$

where

$$\chi_{13}(t, \varepsilon, \theta) = \sum\limits_{\substack{n=-\infty \\ (n \neq m_{13})}}^{\infty} \frac{\Gamma_n[p_{13}(t, \varepsilon, \theta)]}{i(n - m_{13})\varphi(t, \varepsilon)} \, e^{in\theta}, \qquad (27)$$

the system (21) has a family of the 2π-periodic on θ solutions:

$$\xi_{13}^0(t, \varepsilon, \theta) = \chi_{13}(t, \varepsilon, \theta) + M_{13}(t, \varepsilon)e^{im_{13}\theta},$$

$$\xi_{12}^0(t, \varepsilon, \theta) = \chi_{12}(t, \varepsilon, \theta) - a_{12}(t, \varepsilon)e^{im_{13}\theta} + M_{12}(t, \varepsilon)e^{im_{12}\theta},$$

$$\xi_{23}^0(t, \varepsilon, \theta) = \chi_{23}(t, \varepsilon, \theta) + a_{23}(t, \varepsilon)e^{im_{13}\theta} + M_{23}(t, \varepsilon)e^{im_{23}\theta},$$

where

$$\chi_{12}(t, \varepsilon, \theta) = \sum\limits_{\substack{n=-\infty \\ (n \neq m_{12})}}^{\infty} \frac{\Gamma_n[p_{12}(t, \varepsilon, \theta) - b_{32}(t, \varepsilon)\chi_{13}(t, \varepsilon, \theta)]}{i(n - m_{12})\varphi(t, \varepsilon)} \, e^{in\theta},$$

$$a_{12}(t,\varepsilon) = \frac{b_{32}(t,\varepsilon)M_{13}(t,\varepsilon)}{i(m_{13}-m_{12})\varphi(t,\varepsilon)},$$

$$\chi_{23}(t,\varepsilon,\theta) = \sum_{\substack{n=-\infty \\ (n\neq m_{23})}}^{\infty} \frac{\Gamma_n[p_{23}(t,\varepsilon,\theta)+b_{21}(t,\varepsilon)\chi_{13}(t,\varepsilon,\theta)]}{i(n-m_{23})\varphi(t,\varepsilon)} \, e^{in\theta},$$

$$a_{23}(t,\varepsilon) = \frac{b_{32}(t,\varepsilon)M_{13}(t,\varepsilon)}{i(m_{13}-m_{23})\varphi(t,\varepsilon)},$$

and $M_{13}(t,\varepsilon)$, $M_{12}(t,\varepsilon)$, $M_{23}(t,\varepsilon)$ – the function from the class $S(m;\varepsilon_0)$, which defined from the next system of the equations:

$$R_1^{(1)}(t,\varepsilon,M_{12},M_{13},M_{23}) = \int\limits_0^{2\pi} F_{13}\left(t,\varepsilon,\theta,\xi_{12}^0,\xi_{13}^0,\xi_{23}^0\right) e^{-im_{13}\theta}d\theta = 0,$$

$$R_2^{(1)}(t,\varepsilon,M_{12},M_{13},M_{23}) = \int\limits_0^{2\pi} \left(F_{12}\left(t,\varepsilon,\theta,\xi_{12}^0,\xi_{13}^0,\xi_{23}^0\right) - b_{32}(t,\varepsilon)\chi_{13}(t,\varepsilon,\theta)\right) e^{-im_{12}\theta}d\theta = 0,$$

$$R_3^{(1)}(t,\varepsilon,M_{12},M_{13},M_{23}) = \int\limits_0^{2\pi} \left(F_{23}\left(t,\varepsilon,\theta,\xi_{12}^0,\xi_{13}^0,\xi_{23}^0\right) + b_{21}(t,\varepsilon)\chi_{13}(t,\varepsilon,\theta)\right) e^{-im_{23}\theta}d\theta = 0.$$

$$(28)$$

We assume, that the system (28) has a solution $M_{jk}(t,\varepsilon)$, such that

$$\inf_{G(\varepsilon_0)} \left| \det \frac{\partial\left(R_1^{(1)},R_2^{(1)},R_3^{(1)}\right)}{\partial(M_{12},M_{13},M_{23})} \right| > 0. \tag{29}$$

Then, in accordance with the small parameter method, all systems (23) will have a solution, belongs to the class $F(m;\varepsilon_0;\theta)$. Consequently, the functions $\xi_{jk}^*(t,\varepsilon,\theta,\mu)$ in (20) will also belongs to the class $F(m;\varepsilon_0;\theta)$.

Thus, the functions $\xi_{jk}^*(t,\varepsilon,\theta,\mu)$ in the theorem are defined, and the substitution

$$\psi_{jk} = \xi_{jk}^*(t,\varepsilon,\theta,\mu) + \psi_{jk}^1 \ (j<k) \tag{30}$$

leads the system (17) to the kind:

$$\frac{d\psi^1}{dt} = \left(A_1(t,\varepsilon) + \sum_{l=1}^{q} K_l(t,\varepsilon,\theta)\mu^l\right)\psi^1 + \varepsilon g^1(t,\varepsilon,\theta,\mu) + \mu^{2q}c^1(t,\varepsilon,\theta,\mu) +$$

$$+ \mu^{q+1}L^1(t,\varepsilon,\theta,\mu)\psi^1 + \mu\Psi^1(t,\varepsilon,\theta,\psi^1,\mu), \tag{31}$$

where $\psi^1 = \mathrm{colon}(\psi^1_{12}, \psi^1_{13}, \psi^1_{23})$, elements of the matrices K_l, L^1 $(l = \overline{1,q})$ and the vector c^1 belongs to the class $F(m; \varepsilon_0; \theta)$, and the elements of vector g^1 belongs to the class $F(m-1; \varepsilon_0; \theta)$. The components of the vector-function Ψ^1 are the polynomials in respect to elements of vector ψ^1 with coefficients from the class $F(m; \varepsilon_0; \theta)$, and containing the terms not lover second order in respect theese variables.

Based on the lemma, using the transformation of kind

$$\psi^1 = \left(E + \sum_{l=1}^{q} \Psi_l(t, \varepsilon, \theta)\mu_l \right) \sigma \tag{32}$$

we will lead the system (31) to the kind (25).

Case 2. $m_{12} = m_{13}, m_{13} \neq m_{23}$.

In this case under the conditions

$$\int\limits_0^{2\pi} p_{13}(t, \varepsilon, \theta)\ e^{-im_{13}\theta}d\theta = 0,$$

$$\int\limits_0^{2\pi} (p_{23}(t, \varepsilon, \theta) + b_{21}(t, \varepsilon)\chi_{13}(t, \varepsilon, \theta))\ e^{-im_{23}\theta}d\theta = 0, \tag{33}$$

where $\chi_{13}(t, \varepsilon, \theta)$ are defined by formula (27), the system (21) has a family of the 2π-periodic on θ solutions:

$$\xi^0_{13}(t, \varepsilon, \theta) = \chi_{13}(t, \varepsilon, \theta) + M_{13}(t, \varepsilon)e^{im_{13}\theta},$$

$$\xi^0_{12}(t, \varepsilon, \theta) = \sum_{\substack{n=-\infty \\ (n \neq m_{12})}}^{\infty} \frac{\Gamma_n[p_{12}(t, \varepsilon, \theta) - b_{32}(t, \varepsilon)(\chi_{13}(t, \varepsilon, \theta) + M_{13}(t, \varepsilon)e^{im_{13}\theta}]}{i(n - m_{12})\varphi(t, \varepsilon)}\ e^{in\theta}+,$$

$$+M_{12}(t, \varepsilon)e^{im_{12}\theta},$$

$$\xi^0_{23}(t, \varepsilon, \theta) = \sum_{\substack{n=-\infty \\ (n \neq m_{23})}}^{\infty} \frac{\Gamma_n[p_{12}(t, \varepsilon, \theta) + b_{21}(t, \varepsilon)(\chi_{13}(t, \varepsilon, \theta) + M_{13}(t, \varepsilon)e^{im_{13}\theta}]}{i(n - m_{23})\varphi(t, \varepsilon)}\ e^{in\theta}+$$

$$+M_{23}(t, \varepsilon)e^{im_{23}\theta},$$

where $\chi_{13}(t, \varepsilon, \theta)$ are defined by formula (27),

$$M_{13}(t, \varepsilon) = \frac{1}{2\pi b_{32}(t, \varepsilon)} \int\limits_0^{2\pi} (p_{12}(t, \varepsilon, \theta) - b_{32}(t, \varepsilon)\chi(t, \varepsilon, \theta))e^{-im_{12}\theta}d\theta,$$

and $M_{12}(t,\varepsilon)$, $M_{23}(t,\varepsilon)$ – functions of class $S(m;\varepsilon_0)$, which defined from the next system of equations:

$$R_1^{(2)}(t,\varepsilon,M_{12},M_{23}) = \int\limits_0^{2\pi} F_{13}\left(t,\varepsilon,\theta,\xi_{12}^0,\xi_{13}^0,\xi_{23}^0\right) e^{-im_{13}\theta}d\theta = 0,$$

$$R_2^{(2)}(t,\varepsilon,M_{12},M_{23}) = \int\limits_0^{2\pi} \left(F_{23}\left(t,\varepsilon,\theta,\xi_{12}^0,\xi_{13}^0,\xi_{23}^0\right) + b_{21}(t,\varepsilon)\chi_{13}(t,\varepsilon,\theta)\right) e^{-im_{23}\theta}d\theta = 0.$$

$$(34)$$

We assume, that the system (34) has a solution $M_{12}(t,\varepsilon)$, $M_{23}(t,\varepsilon)$ such that

$$\inf_{G(\varepsilon_0)} \left| \det \frac{\partial\left(R_1^{(2)}, R_2^{(2)}\right)}{\partial(M_{12},M_{23})} \right| > 0. \tag{35}$$

Further reasonings are the same as in case 1.

Case 3. $m_{23} = m_{13}$, $m_{12} \neq m_{23}$.

In this case under the conditions

$$\int\limits_0^{2\pi} p_{13}(t,\varepsilon,\theta)\, e^{-im_{13}\theta}d\theta = 0,$$

$$(36)$$

$$\int\limits_0^{2\pi} \left(p_{12}(t,\varepsilon,\theta) - b_{32}(t,\varepsilon)\chi_{13}(t,\varepsilon,\theta)\right) e^{-im_{12}\theta}d\theta = 0,$$

where $\chi_{13}(t,\varepsilon,\theta)$ are defined by formula (27), the system (21) has a family of the 2π-periodic on θ solutions:

$$\xi_{13}^0(t,\varepsilon,\theta) = \chi_{13}(t,\varepsilon,\theta) + M_{13}(t,\varepsilon)e^{im_{13}\theta},$$

$$\xi_{12}^0(t,\varepsilon,\theta) = \sum_{\substack{n=-\infty \\ (n\neq m_{12})}}^{\infty} \frac{\Gamma_n[p_{12}(t,\varepsilon,\theta) - b_{32}(t,\varepsilon)(\chi_{13}(t,\varepsilon,\theta) + M_{13}(t,\varepsilon)e^{im_{13}\theta}]}{i(n-m_{12})\varphi(t,\varepsilon)} e^{in\theta} +,$$

$$+M_{12}(t,\varepsilon)e^{im_{12}\theta},$$

$$\xi_{23}^0(t,\varepsilon,\theta) = \sum_{\substack{n=-\infty \\ (n\neq m_{23})}}^{\infty} \frac{\Gamma_n[p_{23}(t,\varepsilon,\theta) + b_{21}(t,\varepsilon)(\chi_{13}(t,\varepsilon,\theta) + M_{13}(t,\varepsilon)e^{im_{13}\theta}]}{i(n-m_{23})\varphi(t,\varepsilon)} e^{in\theta} +$$

$$+M_{23}(t,\varepsilon)e^{im_{23}\theta},$$

where $\chi_{13}(t,\varepsilon,\theta)$ are defined by formula (27),

$$M_{13}(t,\varepsilon) = -\frac{1}{2\pi b_{21}(t,\varepsilon)} \int\limits_0^{2\pi} (p_{23}(t,\varepsilon,\theta) + b_{21}(t,\varepsilon)\chi(t,\varepsilon,\theta))e^{-im_{23}\theta}d\theta,$$

and $M_{12}(t,\varepsilon)$, $M_{23}(t,\varepsilon)$ – functions of class $S(m;\varepsilon_0)$, which defined from the next system of equations:

$$R_1^{(3)}(t,\varepsilon,M_{12},M_{23}) = \int\limits_0^{2\pi} F_{13}\left(t,\varepsilon,\theta,\xi_{12}^0,\xi_{13}^0,\xi_{23}^0\right) e^{-im_{13}\theta}d\theta = 0,$$

$$R_2^{(3)}(t,\varepsilon,M_{12},M_{23}) =$$

$$= \int\limits_0^{2\pi} \left(F_{12}\left(t,\varepsilon,\theta,\xi_{12}^0,\xi_{13}^0,\xi_{23}^0\right) - b_{32}(t,\varepsilon)\left(\chi_{13}(t,\varepsilon,\theta) + M_{13}(t,\varepsilon)e^{im_{13}\theta}\right)\right) e^{-im_{12}\theta}d\theta = 0.$$

$$(37)$$

We assume, that the system (37) has a solution $M_{12}(t,\varepsilon)$, $M_{23}(t,\varepsilon)$ such that

$$\inf_{G(\varepsilon_0)} \left| \det \frac{\partial\left(R_1^{(3)}, R_2^{(3)}\right)}{\partial(M_{12}, M_{23})} \right| > 0. \qquad (38)$$

Further reasonings are the same as in case 1.

Case 4. $m_{12} = m_{23}$, $m_{12} \neq m_{13}$.

In this case under the conditions

$$\int\limits_0^{2\pi} p_{13}(t,\varepsilon,\theta)\, e^{-im_{13}\theta}d\theta = 0,$$

$$\int\limits_0^{2\pi} (p_{12}(t,\varepsilon,\theta) - b_{32}(t,\varepsilon)\chi_{13}(t,\varepsilon,\theta))\, e^{-im_{12}\theta}d\theta = 0, \qquad (39)$$

$$\int\limits_0^{2\pi} (p_{23}(t,\varepsilon,\theta) = b_{21}(t,\varepsilon)\chi_{13}(t,\varepsilon,\theta))\, e^{-im_{23}\theta}d\theta = 0,$$

where $\chi_{13}(t,\varepsilon,\theta)$ are defined by formula (27), the system (21) has a family of the 2π-periodic on θ solutions:

$$\xi_{13}^0(t,\varepsilon,\theta) = \chi_{13}(t,\varepsilon,\theta) + M_{13}(t,\varepsilon)e^{im_{13}\theta},$$

$$\xi_{12}^0(t,\varepsilon,\theta) = \sum_{\substack{n=-\infty \\ (n \neq m_{12})}}^{\infty} \frac{\Gamma_n[p_{12}(t,\varepsilon,\theta) - b_{32}(t,\varepsilon)(\chi_{13}(t,\varepsilon,\theta) + M_{13}(t,\varepsilon)e^{im_{13}\theta}]}{i(n-m_{12})\varphi(t,\varepsilon)} \, e^{in\theta} +,$$

$$+M_{12}(t,\varepsilon)e^{im_{12}\theta},$$

$$\xi_{23}^0(t,\varepsilon,\theta) = \sum_{\substack{n=-\infty \\ (n \neq m_{23})}}^{\infty} \frac{\Gamma_n[p_{23}(t,\varepsilon,\theta) + b_{21}(t,\varepsilon)(\chi_{13}(t,\varepsilon,\theta) + M_{13}(t,\varepsilon)e^{im_{13}\theta}]}{i(n-m_{23})\varphi(t,\varepsilon)} \, e^{in\theta} +$$

$$+M_{23}(t,\varepsilon)e^{im_{23}\theta},$$

where $\chi_{13}(t,\varepsilon,\theta)$ are defined by formula (27), and $M_{12}(t,\varepsilon)$, $M_{13}(t,\varepsilon)$ $M_{23}(t,\varepsilon)$ – functions of class $S(m;\varepsilon_0)$, which defined from the next system of equations:

$$R_1^{(4)}(t,\varepsilon,M_{12},M_{13},M_{23}) = \int_0^{2\pi} F_{13}\left(t,\varepsilon,\theta,\xi_{12}^0,\xi_{13}^0,\xi_{23}^0\right) e^{-im_{13}\theta} d\theta = 0,$$

$$R_2^{(4)}(t,\varepsilon,M_{12},M_{13},M_{23}) =$$

$$= \int_0^{2\pi} \left(F_{12}\left(t,\varepsilon,\theta,\xi_{12}^0,\xi_{13}^0,\xi_{23}^0\right) - b_{32}(t,\varepsilon)\chi_{13}(t,\varepsilon,\theta)\right) e^{-im_{12}\theta} d\theta = 0, \qquad (40)$$

$$R_3^{(4)}(t,\varepsilon,M_{12},M_{13},M_{23}) =$$

$$= \int_0^{2\pi} \left(F_{23}\left(t,\varepsilon,\theta,\xi_{12}^0,\xi_{13}^0,\xi_{23}^0\right) + b_{21}(t,\varepsilon)\chi_{13}(t,\varepsilon,\theta)\right) e^{-im_{23}\theta} d\theta = 0,$$

We assume, that the system (37) has a solution $M_{12}(t,\varepsilon)$, $M_{13}(t,\varepsilon)$ $M_{23}(t,\varepsilon)$ such that

$$\inf_{G(\varepsilon_0)} \left| \det \frac{\partial\left(R_1^{(4)}, R_2^{(4)}, R_3^{(4)}\right)}{\partial(M_{12}, M_{13}, M_{23})} \right| > 0. \qquad (41)$$

Further reasonings are the same as in case 1.

Case 5. $m_{12} = m_{13} = m_{23} = m$.

In this case under the conditions

$$\int_0^{2\pi} p_{13}(t,\varepsilon,\theta) \, e^{-im\theta} d\theta = 0,$$

$$\qquad (42)$$

$$\int_0^{2\pi} \left(\frac{p_{12}(t,\varepsilon,\theta)}{b_{32}(t,\varepsilon)} + \frac{p_{23}(t,\varepsilon,\theta)}{b_{21}(t,\varepsilon)}\right) e^{-im\theta} d\theta = 0,$$

the system (21) has a family of the 2π-periodic on θ solutions:

$$\xi_{13}^0(t,\varepsilon,\theta) = \chi_{13}(t,\varepsilon,\theta) + M_{13}(t,\varepsilon)e^{im\theta},$$

$$\xi_{12}^0(t,\varepsilon,\theta) = \sum_{\substack{n=-\infty \\ (n\neq m)}}^{\infty} \frac{\Gamma_n[p_{12}(t,\varepsilon,\theta) - b_{32}(t,\varepsilon)(\chi_{13}(t,\varepsilon,\theta) + M_{13}(t,\varepsilon)e^{im\theta}]}{i(n-m)\varphi(t,\varepsilon)} e^{in\theta}+,$$

$$+M_{12}(t,\varepsilon)e^{im\theta},$$

$$\xi_{23}^0(t,\varepsilon,\theta) = \sum_{\substack{n=-\infty \\ (n\neq m)}}^{\infty} \frac{\Gamma_n[p_{23}(t,\varepsilon,\theta) + b_{21}(t,\varepsilon)(\chi_{13}(t,\varepsilon,\theta) + M_{13}(t,\varepsilon)e^{im_{13}\theta}]}{i(n-m)\varphi(t,\varepsilon)} e^{in\theta}+$$

$$+M_{23}(t,\varepsilon)e^{im\theta},$$

where $\chi_{13}(t,\varepsilon,\theta)$ are defined by formula (27),

$$M_{13}(t,\varepsilon) = -\frac{1}{2\pi b_{21}(t,\varepsilon)} \int_0^{2\pi} (p_{23}(t,\varepsilon,\theta) + b_{21}(t,\varepsilon)\chi(t,\varepsilon,\theta))e^{-im\theta}d\theta,$$

and $M_{12}(t,\varepsilon)$, $M_{23}(t,\varepsilon)$ – functions of class $S(m;\varepsilon_0)$, which defined from the next system of equations:

$$R_1^{(5)}(t,\varepsilon,M_{12},M_{23}) = \int_0^{2\pi} F_{13}\left(t,\varepsilon,\theta,\xi_{12}^0,\xi_{13}^0,\xi_{23}^0\right)e^{-im\theta}d\theta = 0,$$

$$R_2^{(5)}(t,\varepsilon,M_{12},M_{23}) = \tag{43}$$

$$= \int_0^{2\pi} \left(F_{23}\left(t,\varepsilon,\theta,\xi_{12}^0,\xi_{13}^0,\xi_{23}^0\right) + b_{21}(t,\varepsilon)\chi_{13}(t,\varepsilon,\theta)\right)e^{-im\theta}d\theta = 0,$$

We assume, that the system (37) has a solution $M_{12}(t,\varepsilon)$, $M_{23}(t,\varepsilon)$ such that

$$\inf_{G(\varepsilon_0)} \left|\det \frac{\partial\left(R_1^{(5)}, R_2^{(5)}\right)}{\partial(M_{12}, M_{23})}\right| > 0. \tag{44}$$

Further reasonings are the same as in case 1.

4.   Conclusion

Thus, for the system (2) the conditions of the existence of the transformation with coefficients are represented as an absolutely and uniformly convergent Fourier-series with slowly varying coefficients and frequency, which leads it to triangular kind, are obtained in the resonant cases.

Щоголев С. А.

Про зведення лінійної системи диференціальних рівнянь з коеффіцієнтами коливного типу до трикутного виглядув резонансному випадку

Резюме

Для лінійної однорідної диференціальної системи, коефіцієнти якої зображувані у вигляді абсолютно та рівномірно збіжних рядів Фур'є з повільно змінними коефіцієнтами та частотою, одержано умови існування перетворення, що приводить цю систему до трикутного вигляду в резонансному випадку

Ключові слова: лінійні диференціальні системи, трикутний вигляд, ряди Фур'є, повільно змінні параметри.

Щёголев С. А.

О приведении динейной системы дифференциальных уравнений с коэффициентами осциллирующего типа к треугольному виду в резонансном случае

Резюме

Для линейной однородной дифференциальной системы, коэффициенты которой представимы в виде абсолютно и равномерно сходящихся рядов Фурье с медленно меняющимися коэффициентами и частотой, получены условия существования преобразования, приводящего эту систему к треугольному виду в нерезонансном случае.

Ключевые слова: линейные дифференциальные системы, треугольный вид, ряды Фурье, медленно меняющиеся параметры.

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