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MINIMAL GENERATING SET OF THE COMMUTATOR SUBGROUP OF SYLOW 2-SUBGROUPS OF ALTERNATING GROUP AND ITS STRUCTURE

The size of a minimal generating set for the commutator subgroup of Sylow 2-subgroups of alternating group is found. The structure of commutator subgroup of Sylow 2-subgroups of the alternating group A_{2^k} is investigated.

It is shown that $(Syl_2 A_{2^k})^2 = Syl_2' A_{2^k}$, $k > 2$.

It is proved that the commutator length of an arbitrary element of the iterated wreath product of cyclic groups C_{p_i}, $p_i \in \mathbb{N}$ equals to 1. The commutator width of direct limit of wreath product of cyclic groups is found. This paper presents upper bounds of the commutator width $(cw(G))$ [1] of a wreath product of groups.

A recursive presentation of Sylows 2-subgroups $Syl_2(A_{2^k})$ of A_{2^k} is introduced. As a result the short proof that the commutator width of Sylow 2-subgroups of alternating group A_{2^k}, permutation group S_{2^k} and Sylow p-subgroups of $Syl_2 A_{p^k}$ $(Syl_2 S_{p^k})$ are equal to 1 is obtained.

A commutator width of permutational wreath product $B \wr C_n$ is investigated. An upper bound of the commutator width of permutational wreath product $B \wr C_n$ for an arbitrary group B is found.

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1. Introduction

Meldrum J. [2] briefly considered one form of commutators of the wreath product $A \wr B$. In order to obtain a more detailed description of this form, we take into account the commutator width $(cw(G))$ as presented in work of Muranov A. [1].

As well known the first example of a group G with $cw(G) > 1$ was given by Fite [4]. The smallest finite examples of such groups are groups of order 96, there's two of them, nonisomorphic to each other, were given by Guralnick [23].

We deduce an estimation for commutator width of wreath product of groups $C_n \wr B$ taking in consideration a $cw(B)$ of passive group B. A form of commutators of wreath product $A \wr B$ that was shortly considered in [2]. The form of commutator presentation [2] is proposed by us as wreath recursion [9] and commutator width of it was studied. We impose more weak condition

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on the presentation of wreath product commutator then it was imposed by J. Meldrum.

In this paper we continue a researches which was stared in [16; 17]. We find a minimal generating set and the structure for commutator subgroup of $Syl_2 A_{2^k}$.

A research of commutator-group serve to decision of inclusion problem [5] for elements of $Syl_2 A_{2^k}$ in its derived subgroup $(Syl_2 A_{2^k})'$. It was known that, the commutator width of iterated wreath products of nonabelian finite simple groups is bounded by an absolute constant [3; 4]. But it was not proven that commutator subgroup of $\mathop{\wr}\limits_{i=1}^{k} \mathcal{C}_{p_i}$ consists of commutators. We generalize the passive group of this wreath product to any group B instead of only wreath product of cyclic groups and obtain an exact commutator width.

Also we are going to prove that the commutator width of Sylows p-subgroups of symmetric and alternating groups $p \geq 2$ is 1.

The (permutational) wreath product $H \wr G$ is the semidirect product $H^X \leftthreetimes G$, where G acts on the direct power H^X by the respective permutations of the direct factors. The group C_p or (C_p, X) is equipped with a natural action by the left shift on $X = \{1, \dots, p\}$, $p \in \mathbb{N}$. As well known that a wreath product of permutation groups is associative construction.

The multiplication rule of automorphisms g, h which presented in form of the wreath recursion [6] $g = (g_{(1)}, g_{(2)}, \dots, g_{(d)})\sigma_g$, $h = (h_{(1)}, h_{(2)}, \dots, h_{(d)})\sigma_h$, is given by the formula:

$$g \cdot h = (g_{(1)} h_{(\sigma_g(1))}, g_{(2)} h_{(\sigma_g(2))}, \dots, g_{(d)} h_{(\sigma_g(d))})\sigma_g \sigma_h.$$

We define σ as $(1, 2, \dots, p)$ where p is defined by context.

The set X^* is naturally a vertex set of a regular rooted tree, i.e. a connected graph without cycles and a designated vertex v_0 called the root, in which two words are connected by an edge if and only if they are of form v and vx, where $v \in X^*$, $x \in X$. The set $X^n \subset X^*$ is called the n-th level of the tree X^* and $X^0 = \{v_0\}$. We denote by v_{ji} the vertex of X^j, which has the number i. Note that the unique vertex $v_{k,i}$ corresponds to the unique word v in alphabet X. For every automorphism $g \in AutX^*$ and every word $v \in X^*$ define the section (state) $g_{(v)} \in AutX^*$ of g at v by the rule: $g_{(v)}(x) = y$ for $x, y \in X^*$ if and only if $g(vx) = g(v)y$. The subtree of X^* induced by the set of vertices $\cup_{i=0}^{k} X^i$ is denoted by $X^{[k]}$. The restriction of the action of an automorphism $g \in AutX^*$ to the subtree $X^{[l]}$ is denoted by $g_{(v)}|_{X^{[l]}}$. A restriction $g_{(v_{ij})}|_{X^{[1]}}$ is called the vertex permutation (v.p.) of g in a vertex v_{ij} and denoted by g_{ij}. We call the endomorphism $\alpha|_v$ restriction of g in a vertex v [6]. For example, if $|X| = 2$ then we just have to distinguish active vertices, i.e., the vertices for which $\alpha|_v$ is non-trivial.

Let us label every vertex of X^l, $0 \leq l < k$ by sign 0 or 1 in relation to state of v.p. in it. Obtained by such way a vertex-labeled regular tree is an element of $AutX^{[k]}$. All undeclared terms are from [7; 8].

Let us make some notations. For brevity, in form of wreath recursion we write a commutator as $[a, b] = aba^{-1}b^{-1}$ that is inverse to $a^{-1}b^{-1}ab$. That does not reduce the generality of our reasoning. Since for convenience the commutator of two group elements a and b is denoted by $[a, b] = aba^{-1}b^{-1}$, conjugation by an element b as

$$a^b = bab^{-1},$$

We define G_k and B_k recursively i.e.

$$B_1 = C_2,\ B_k = B_{k-1} \wr C_2 \text{ for } k > 1,$$
$$G_1 = \langle e \rangle,\ G_k = \{(g_1, g_2)\pi \in B_k \mid g_1 g_2 \in G_{k-1}\} \text{ for } k > 1.$$

Note that $B_k = \overset{k}{\underset{i=1}{\wr}} C_2$.

The commutator length of an element g of the derived subgroup of a group G, denoted *clG(g)*, is the minimal n such that there exist elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ in G such that $g = [x_1, y_1] \ldots [x_n, y_n]$. The commutator length of the identity element is 0. The commutator width of a group G, denoted $cw(G)$, is the maximum of the commutator lengths of the elements of its derived subgroup $[G, G]$. We denote by $d(G)$ the minimal number of generators of the group G.

2. Main Results

Commutator width of Sylow 2-subgroups of A_{2^k} and S_{2^k}.

The following Lemma imposes the Corollary 4.9 of [2] and it will be deduced from the corollary 4.9 with using in presentation elements in the form of wreath recursion.

Lemma 1. *An element of form $(r_1, \ldots, r_{p-1}, r_p) \in W' = (B \wr C_p)'$ iff product of all r_i (in any order) belongs to B', where $p \in \mathrm{N}$, $p \geq 2$.*

Proof. More details of our argument may be given as follows.

$$w = (r_1, r_2, \ldots, r_{p-1}, r_p),$$

where $r_i \in B$. If we multiply elements from a tuple $(r_1, \ldots, r_{p-1}, r_p)$, where $r_i = h_i g_{a(i)} h^{-1}_{ab(i)} g^{-1}_{aba^{-1}(i)}$, $h,\ g \in B$ and $a, b \in C_p$, then we get a product

$$x = \prod_{i=1}^{p} r_i = \prod_{i=1}^{p} h_i g_{a(i)} h^{-1}_{ab(i)} g^{-1}_{aba^{-1}(i)} \in B', \tag{1}$$

where x is a product of corespondent commutators. Therefore, we can write $r_p = r_{p-1}^{-1} \dots r_1^{-1} x$. We can rewrite element $x \in B'$ as the product $x = \prod_{j=1}^{m} [f_j, g_j]$, $m \leq cw(B)$.

Note that we impose more weak condition on the product of all r_i to belongs to B' then in Definition 4.5. of form $P(L)$ in [2], where the product of all r_i belongs to a subgroup L of B such that $L > B'$.

In more detail deducing of our representation constructing can be reported in following way. If we multiply elements having form of a tuple $(r_1, \dots, r_{p-1}, r_p)$, where $r_i = h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1}$, h, $g \in B$ and $a, b \in C_p$, then in case $cw(B) = 0$ we obtain a product

$$\prod_{i=1}^{p} r_i = \prod_{i=1}^{p} h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1} \in B'. \tag{2}$$

Note that if we rearrange elements in (1) as

$$h_1 h_1^{-1} g_1 g_2^{-1} h_2 h_2^{-1} g_1 g_2^{-1} \dots h_p h_p^{-1} g_p g_p^{-1},$$

then by the reason of such permutations we obtain a product of corespondent commutators. Therefore, following equality holds true

$$\prod_{i=1}^{p} h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1} = \prod_{i=1}^{p} h_i g_i h_i^{-1} g_i^{-1} x_0 = \prod_{i=1}^{p} h_i h_i^{-1} g_i g_i^{-1} x \in B', \tag{3}$$

where x_0, x are a products of corespondent commutators. Therefore,

$$(r_1, \dots, r_{p-1}, r_p) \in W' \text{ iff } r_{p-1} \cdot \ldots \cdot r_1 \cdot r_p = x \in B'. \tag{4}$$

Thus, one element from states of wreath recursion $(r_1, \dots, r_{p-1}, r_p)$ depends on rest of r_i. This dependence contribute that the product $\prod_{j=1}^{p} r_j$ for an arbitrary sequence $\{r_j\}_{j=1}^{p}$ belongs to B'. Thus, r_p can be expressed as:

$$r_p = r_1^{-1} \cdot \ldots \cdot r_{p-1}^{-1} x.$$

Denote a j-th tuple, which consists of a wreath recursion elements, by $(r_{j_1}, r_{j_2}, \dots, r_{j_p})$. Closedness by multiplication of the set of forms $(r_1, \dots, r_{p-1}, r_p) \in W = (B \wr C_p)'$ follows from

$$\prod_{j=1}^{k} (r_{j1} \dots r_{jp-1} r_{jp}) = \prod_{j=1}^{k} \prod_{i=1}^{p} r_{j_i} = R_1 R_2 \dots R_k \in B', \tag{5}$$

where r_{ji} is i-th element from the tuple number j, $R_j = \prod_{i=1}^{p} r_{ji}$, $1 \le j \le k$. As it was shown above $R_j = \prod_{i=1}^{p-1} r_{ji} \in B'$. Therefore, the product (5) of R_j, $j \in \{1, ..., k\}$ which is similar to the product mentioned in [2], has the property $R_1 R_2 ... R_k \in B'$ too, because of B' is subgroup. Thus, we get a product of form (1) and the similar reasoning as above are applicable.

Let us prove the sufficiency condition. If the set K of elements satisfying the condition of this theorem, that all products of all r_i, where every i occurs in this forms once, belong to B', then using the elements of form

$$(r_1, e, ..., e, r_1^{-1}), ..., (e, e, ..., e, r_i, e, r_i^{-1}), ..., \\ (e, e, ..., e, r_{p-1}, r_{p-1}^{-1}), (e, e, ..., e, r_1 r_2 \cdot ... \cdot r_{p-1})$$

we can express any element of form $(r_1, \ldots, r_{p-1}, r_p) \in W = (B \wr C_p)'$. We need to prove that in such way we can express all element from W and only elements of W. The fact that all elements can be generated by elements of K follows from randomness of choice every r_i, $i < p$ and the fact that equality (1) holds so construction of r_p is determined.

Lemma 2. *For any group B and integer $p \ge 2$ if $w \in (B \wr C_p)'$ then w can be represented as the following wreath recursion*

$$w = (r_1, r_2, \ldots, r_{p-1}, r_1^{-1} \ldots r_{p-1}^{-1} \prod_{j=1}^{k} [f_j, g_j]),$$

where $r_1, \ldots, r_{p-1}, f_j, g_j \in B$ and $k \le cw(B)$.

Proof. According to Lemma 1 we have the following wreath recursion

$$w = (r_1, r_2, \ldots, r_{p-1}, r_p),$$

where $r_i \in B$ and $r_{p-1} r_{p-2} \ldots r_2 r_1 r_p = x \in B'$. Therefore we can write $r_p = r_1^{-1} \ldots r_{p-1}^{-1} x$. We also can rewrite element $x \in B'$ as product of commutators $x = \prod_{j=1}^{k} [f_j, g_j]$ where $k \le cw(B)$.

Lemma 3. *For any group B and integer $p \ge 2$ if $w \in (B \wr C_p)'$ is defined by the following wreath recursion*

$$w = (r_1, r_2, \ldots, r_{p-1}, r_1^{-1} \ldots r_{p-1}^{-1} [f, g]),$$

where $r_1, \ldots, r_{p-1}, f, g \in B$ then we can represent w as the following commutator

$$w = [(a_{1,1}, \ldots, a_{1,p})\sigma, (a_{2,1}, \ldots, a_{2,p})],$$

where

$$\begin{aligned}
&a_{1,i} = e,\ \textit{for } 1 \le i \le p-1\ ,\\
&a_{2,1} = (f^{-1})^{r_1^{-1} \dots r_{p-1}^{-1}},\\
&a_{2,i} = r_{i-1} a_{2,i-1},\ \textit{for } 2 \le i \le p,\\
&a_{1,p} = g^{a_{2,p}^{-1}}.
\end{aligned}$$

Proof. Let us to consider the following commutator

$$\begin{aligned}
\kappa &= (a_{1,1}, \dots, a_{1,p})\sigma \cdot (a_{2,1}, \dots, a_{2,p}) \cdot (a_{1,p}^{-1}, a_{1,1}^{-1}, \dots, a_{1,p-1}^{-1})\sigma^{-1} \cdot (a_{2,1}^{-1}, \dots, a_{2,p}^{-1})\\
&= (a_{3,1}, \dots, a_{3,p}),
\end{aligned}$$

where

$$a_{3,i} = a_{1,i} a_{2,1+(i \bmod p)} a_{1,i}^{-1} a_{2,i}^{-1}.$$

At first we compute the following

$$a_{3,i} = a_{1,i} a_{2,i+1} a_{1,i}^{-1} a_{2,i}^{-1} = a_{2,i+1} a_{2,i}^{-1} = r_i a_{2,i} a_{2,i}^{-1} = r_i,\ \text{for } 1 \le i \le p-1.$$

Then we make some transformation of $a_{3,p}$:

$$\begin{aligned}
a_{3,p} &= a_{1,p} a_{2,1} a_{1,p}^{-1} a_{2,p}^{-1}\\
&= (a_{2,1} a_{2,1}^{-1}) a_{1,p} a_{2,1} a_{1,p}^{-1} a_{2,p}^{-1}\\
&= a_{2,1} [a_{2,1}^{-1}, a_{1,p}] a_{2,p}^{-1}\\
&= a_{2,1} a_{2,p}^{-1} a_{2,p} [a_{2,1}^{-1}, a_{1,p}] a_{2,p}^{-1}\\
&= (a_{2,p} a_{2,1}^{-1})^{-1} [(a_{2,1}^{-1})^{a_{2,p}}, a_{1,p}^{a_{2,p}}]\\
&= (a_{2,p} a_{2,1}^{-1})^{-1} [(a_{2,1}^{-1})^{a_{2,p} a_{2,1}^{-1}}, a_{1,p}^{a_{2,p}}].
\end{aligned}$$

Now we can see that the form of the commutator κ is similar to the form of w.

Let us make the following notation

$$r' = r_{p-1} \dots r_1.$$

We note that from the definition of $a_{2,i}$ for $2 \le i \le p$ it follows that

$$r_i = a_{2,i+1} a_{2,i}^{-1},\ \text{for } 1 \le i \le p-1.$$

Therefore

$$\begin{aligned}
r' &= (a_{2,p} a_{2,p-1}^{-1})(a_{2,p-1} a_{2,p-2}^{-1}) \dots (a_{2,3} a_{2,2}^{-1})(a_{2,2} a_{2,1}^{-1})\\
&= a_{2,p} a_{2,1}^{-1}.
\end{aligned}$$

And then

$$(a_{2,p}a_{2,1}^{-1})^{-1} = (r')^{-1} = r_1^{-1} \dots r_{p-1}^{-1}.$$

And now we compute the following

$$\begin{aligned}(a_{2,1}^{-1})^{a_{2,p}a_{2,1}^{-1}} &= (((f^{-1})^{r_1^{-1} \dots r_{p-1}^{-1}})^{-1})^{r'} = (f^{(r')^{-1}})^{r'} = f,\\ a_{1,p}^{a_{2,p}} &= (g^{a_{2,p}^{-1}})^{a_{2,p}} = g.\end{aligned}$$

Finally we conclude that

$$a_{3,p} = r_1^{-1} \dots r_{p-1}^{-1}[f, g].$$

Thus, the commutator κ is presented exactly in the similar form as w has.

For future using we formulate previous Lemma for the case $p = 2$.

Corollary 1. *For any group B if $w \in (B \wr C_2)'$ is defined by the following wreath recursion*

$$w = (r_1, r_1^{-1}[f, g]),$$

where $r_1, f, g \in B$ then we can represent w as commutator

$$w = [(e, a_{1,2})\sigma, (a_{2,1}, a_{2,2})],$$

where

$$\begin{aligned}a_{2,1} &= (f^{-1})^{r_1^{-1}},\\ a_{2,2} &= r_1 a_{2,1},\\ a_{1,2} &= g^{a_{2,2}^{-1}}.\end{aligned}$$

Lemma 4. *For any group B and integer $p \geq 2$ inequality*

$$cw(B \wr C_p) \leq \max(1, cw(B))$$

holds.

Proof. We can represent any $w \in (B \wr C_p)'$ by Lemma 1 with the following wreath recursion

$$\begin{aligned}w &= (r_1, r_2, \dots, r_{p-1}, r_1^{-1} \dots, r_{p-1}^{-1} \prod_{j=1}^{k} [f_j, g_j])\\ &= (r_1, r_2, \dots, r_{p-1}, r_1^{-1} \dots, r_{p-1}^{-1}[f_1, g_1]) \cdot \prod_{j=2}^{k} [(e, \dots, e, f_j), (e, \dots, e, g_j)],\end{aligned}$$

where $r_1, \dots, r_{p-1}, f_j, g_j \in B$ and $k \leq cw(B)$. Now by the Lemma 3 we can see that w can be represented as a product of $\max(1, cw(B))$ commutators.

Corollary 2. *If* $W = C_{p_k} \wr \ldots \wr C_{p_1}$ *then* $cw(W) = 1$ *for* $k \geq 2$.

Proof. If $B = C_{p_k} \wr C_{p_{k-1}}$ then taking into consideration that $cw(B) > 0$ (because $C_{p_k} \wr C_{p_{k-1}}$ is not commutative group). Since Lemma 4 implies that $cw(C_{p_k} \wr C_{p_{k-1}}) = 1$ then according to the inequality $cw(C_{p_k} \wr C_{p_{k-1}} \wr C_{p_{k-2}}) \leq \max(1, cw(B))$ from Lemma 4 we obtain $cw(C_{p_k} \wr C_{p_{k-1}} \wr C_{p_{k-2}}) = 1$. Analogously if $W = C_{p_k} \wr \ldots \wr C_{p_1}$ and supposition of induction for $C_{p_k} \wr \ldots \wr C_{p_2}$ holds, then using an associativity of a permutational wreath product we obtain from the inequality of Lemma 4 and the equality $cw(C_{p_k} \wr \ldots \wr C_{p_2}) = 1$ that $cw(W) = 1$.

We define our partial ordered set M as the set of all finite wreath products of cyclic groups. We make of use directed set $\mathbb{N}$.

$$H_k = \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i} \tag{6}$$

Moreover, it has already been proved in Corollary 3 that each group of the form $\underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}$ has a commutator width equal to 1, i.e $cw(\underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}) = 1$. A partial order relation will be a subgroup relationship. Define the injective homomorphism $f_{k,k+1}$ from the $\underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}$ into $\underset{i=1}{\overset{k+1}{\wr}} \mathcal{C}_{p_i}$ by mapping a generator of active group $\mathcal{C}_{p_i}$ of H_k in a generator of active group $\mathcal{C}_{p_i}$ of H_{k+1}. In more details the injective homomorphism $f_{k,k+1}$ is defined as $g \mapsto g(e, ..., e)$, where a generator $g \in \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}$, $g(e, ..., e) \in \underset{i=1}{\overset{k+1}{\wr}} \mathcal{C}_{p_i}$.

Therefore this is an injective homomorphism of H_k onto subgroup $\underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}$ of H_{k+1}.

Corollary 3. *The direct limit* $\varinjlim \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}$ *of direct system* $\left\langle f_{k,j}, \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i} \right\rangle$ *has commutator width 1.*

Proof. We make the transition to the direct limit in the direct system $\left\langle f_{k,j}, \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i} \right\rangle$ of injective mappings from chain $e \to \ \ldots \ \to \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i} \to \underset{i=1}{\overset{k+1}{\wr}} \mathcal{C}_{p_i} \to \underset{i=1}{\overset{k+2}{\wr}} \mathcal{C}_{p_i} \to ....$

Since all mappings in chains are injective homomorphisms, it has a trivial kernel. Therefore the transition to a direct limit boundary preserves the property $cw(H) = 1$, because each group H_k from the chain endowed by $cw(H_k) = 1$.

The direct limit of the direct system is denoted by $\varinjlim \wr_{i=1}^{k} \mathcal{C}_{p_i}$ and is defined as disjoint union of the H_k's modulo a certain equivalence relation:

$$\varinjlim \wr_{i=1}^{k} \mathcal{C}_{p_i} = {}^{\coprod_k \wr_{i=1}^{k} \mathcal{C}_{p_i}}/\sim.$$

Since every element g of $\varinjlim \wr_{i=1}^{k} \mathcal{C}_{p_i}$ coincides with a correspondent element from some H_k of direct system, then by the injectivity of the mappings for g the property $cw(\wr_{i=1}^{k} \mathcal{C}_{p_i}) = 1$ also holds. Thus, it holds for the whole $\varinjlim \wr_{i=1}^{k} \mathcal{C}_{p_i}$.

Corollary 4. *For prime p and $k \geq 2$ commutator width $cw(Syl_p(S_{p^k})) = 1$ and for prime $p > 2$ and $k \geq 2$ commutator width $cw(Syl_p(A_{p^k})) = 1$.*

Proof. Since $Syl_p(S_{p^k}) \simeq \wr_{i=1}^{k} C_p$ see [10; 11], then $cw(Syl_p(S_{p^k})) = 1$. As well known in case $p > 2$ we have $Syl_p S_{p^k} \simeq Syl_p A_{p^k}$ see [16; 19], then $cw(Syl_p(A_{p^k})) = 1$.

Proposition 1. *The following inclusion $B'_k < G_k$ holds.*

Proof. Induction on k. For $k = 1$ we have $B'_k = G_k = \{e\}$. Let us fix some $g = (g_1, g_2) \in B'_k$. Then $g_1 g_2 \in B'_{k-1}$ by Lemma 1. As $B'_{k-1} < G_{k-1}$ by induction hypothesis therefore $g_1 g_2 \in G_{k-1}$ and by definition of G_k it follows that $g \in G_k$.

Corollary 5. *The set G_k is a subgroup in the group B_k.*

Proof. According to recursively definition of G_k and B_k, where $G_k = \{(g_1, g_2)\pi \in B_k \mid g_1 g_2 \in G_{k-1}\}$ $k > 1$, G_k is subset of B_k with condition $g_1 g_2 \in G_{k-1}$. It is easy to check the closedness by multiplication elements of G_k with condition $g_1 g_2, h_1 h_2 \in G_{k-1}$ because G_{k-1} is subgroup so $g_1 g_2 h_1 h_2 \in G_{k-1}$ too. A condition of existing inverse be verified trivial.

Lemma 5. *For any $k \geq 1$ we have $|G_k| = |B_k|/2$.*

Proof. Induction on k. For $k = 1$ we have $|G_1| = 1 = |B_1/2|$. Every element $g \in G_k$ can be uniquely write as the following wreath recursion

$$g = (g_1, g_2)\pi = (g_1, g_1^{-1}x)\pi$$

where $g_1 \in B_{k-1}$, $x \in G_{k-1}$ and $\pi \in C_2$. Elements g_1, x and π are independent therefore $|G_k| = 2|B_{k-1}| \cdot |G_{k-1}| = 2|B_{k-1}| \cdot |B_{k-1}|/2 = |B_k|/2$.

Corollary 6. *The group G_k is a normal subgroup in the group B_k i.e. $G_k \lhd B_k$.*

Proof. There exists normal embedding (normal injective monomorphism) $\varphi:\ G_k \to B_k$ [20] such that $G_k \triangleleft B_k$. Indeed, according to Lemma index $|B_k:\ G_k| = 2$ so it is normal subgroup that is quotient subgroup ${}^{B_k}/_{C_2} \simeq G_k$.

Theorem 1. *For any $k \geq 1$ we have $G_k \simeq Syl_2 A_{2^k}$.*

Proof. Group C_2 acts on the set $X = \{1, 2\}$. Therefore we can recursively define sets X^k on which group B_k acts $X^1 = X$, $X^k = X^{k-1} \times X$ for k>1. At first we define $S_{2^k} = Sym(X^k)$ and $A_{2^k} = Alt(X^k)$ for all integer $k \geq 1$. Then $G_k < B_k < S_{2^k}$ and $A_{2^k} < S_{2^k}$.

We already know [16] that $B_k \simeq Syl_2(S_{2^k})$. Since $|A_{2^k}| = |S_{2^k}|/2$ therefore $|Syl_2 A_{2^k}| = |Syl_2 S_{2^k}|/2 = |B_k|/2$. By Lemma 2 it follows that $|Syl_2 A_{2^k}| = |G_k|$. Therefore it is left to show that $G_k < Alt(X^k)$.

Let us fix some $g = (g_1, g_2)\sigma^i$ where $g_1, g_2 \in B_{k-1}$, $i \in \{0, 1\}$ and $g_1 g_2 \in G_{k-1}$. Then we can represent g as follows

$$g = (g_1 g_2, e) \cdot (g_2^{-1}, g_2) \cdot (e, e,)\sigma^i.$$

In order to prove this theorem it is enough to show that

$$(g_1 g_2, e), (g_2^{-1}, g_2), (e, e,)\sigma \in Alt(X^k).$$

Element $(e, e,)\sigma$ just switch letters x_1 and x_2 for all $x \in X^k$. Therefore $(e, e,)\sigma$ is product of $|X^{k-1}| = 2^{k-1}$ transpositions and therefore $(e, e,)\sigma \in Alt(X^k)$.

Elements g_2^{-1} and g_2 have the same cycle type. Therefore elements (g_2^{-1}, e) and (e, g_2) also have the same cycle type. Let us fix the following cycle decompositions

$$(g_2^{-1}, e) = \sigma_1 \cdot \ldots \cdot \sigma_n,$$
$$(e, g_2) = \pi_1 \cdot \ldots \cdot \pi_n.$$

Note that element (g_2^{-1}, e) acts only on letters like x_1 and element (e, g_2) acts only on letters like x_2. Therefore we have the following cycle decomposition

$$(g_2^{-1}, g_2) = \sigma_1 \cdot \ldots \cdot \sigma_n \cdot \pi_1 \cdot \ldots \cdot \pi_n.$$

So, element (g_2^{-1}, g_2) has even number of odd permutations and then $(g_2^{-1}, g_2) \in Alt(X^k)$.

Note that $g_1 g_2 \in G_{k-1}$ and $G_{k-1} = Alt(X^{k-1})$ by induction hypothesis. Therefore $g_1 g_2 \in Alt(X^{k-1})$. As elements $g_1 g_2$ and $(g_1 g_2, e)$ have the same cycle type then $(g_1 g_2, e) \in Alt(X^k)$.

As it was proven by the author in [16] Sylow 2-subgroup has structure $B_{k-1} \ltimes W_{k-1}$, where definition of B_{k-1} is the same that was given in [16].

Recall that it was denoted by W_{k-1} the subgroup of $AutX^{[k]}$ such that has active states only on X^{k-1} and number of such states is even, i.e. $W_{k-1} \lhd St_{G_k}(k-1)$ [6]. It was proven that the size of W_{k-1} is equal to $2^{2^{k-1}-1}$, $k > 1$ and its structure is $(C_2)^{2^{k-1}-1}$. The following structural theorem characterizing the group G_k was proved by us [16].

Theorem 2. *A maximal 2-subgroup of $AutX^{[k]}$ that acts by even permutations on X^k has the structure of the semidirect product $G_k \simeq B_{k-1} \ltimes W_{k-1}$ and isomorphic to $Syl_2 A_{2^k}$.*

Note that W_{k-1} is subgroup of stabilizer of X^{k-1} i.e. $W_{k-1} < St_{AutX^{[k]}}(k-1) \lhd AutX^{[k]}$ and is normal too $W_{k-1} \lhd AutX^{[k]}$, because conjugation keeps a cyclic structure of permutation so even permutation maps in even. Therefore such conjugation induce an automorphism of W_{k-1} and $G_k \simeq B_{k-1} \ltimes W_{k-1}$.

Remark 1. *As a consequence, the structure founded by us in [16] fully consistent with the recursive group representation (which used in this paper) based on the concept of wreath recursion [9].*

Theorem 3. *Elements of B'_k have the following form $B'_k = \{[f, l] \mid f \in B_k, l \in G_k\} = \{[l, f] \mid f \in B_k, l \in G_k\}$.*

Proof. It is enough to show either $B'_k = \{[f, l] \mid f \in B_k, l \in G_k\}$ or $B'_k = \{[l, f] \mid f \in B_k, l \in G_k\}$ because if $f = [g, h]$ then $f^{-1} = [h, g]$.

We prove the proposition by induction on k. For the case $k = 1$ we have $B'_1 = \langle e \rangle$.

Consider case $k > 1$. According to Lemma 2 and Corollary 1 every element $w \in B'_k$ can be represented as

$$w = (r_1, r_1^{-1}[f, g])$$

for some $r_1, f \in B_{k-1}$ and $g \in G_{k-1}$ (by induction hypothesis). By the Corollary 1 we can represent w as commutator of

$$(e, a_{1,2})\sigma \in B_k \text{ and } (a_{2,1}, a_{2,2}) \in B_k,$$

where

$$\begin{aligned} a_{2,1} &= (f^{-1})^{r_1^{-1}}, \\ a_{2,2} &= r_1 a_{2,1}, \\ a_{1,2} &= g^{a_{2,2}^{-1}}. \end{aligned}$$

If $g \in G_{k-1}$ then by the definition of G_k and Corollary 6 we obtain $(e, a_{1,2})\sigma \in G_k$.

Remark 2. *Let us to note that Theorem 3 improve Corollary 4 for the case* $Syl_2 S_{2^k}$.

Proposition 2. *If g is an element of the group B_k then $g^2 \in B'_k$.*

Proof. Induction on k. We note that $B_k = B_{k-1} \wr C_2$. Therefore we fix some element

$$g = (g_1, g_2)\sigma^i \in B_{k-1} \wr C_2,$$

where $g_1, g_2 \in B_{k-1}$ and $i \in \{0, 1\}$. Let us to consider g^2 then two cases are possible:

$$g^2 = (g_1^2, g_2^2) \text{ or } g^2 = (g_1 g_2, g_2 g_1)$$

In second case we consider a product of coordinates $g_1 g_2 \cdot g_2 g_1 = g_1^2 g_2^2 x$. Since according to the induction hypothesis $g_i^2 \in B'_k$, $i \leq 2$ then $g_1 g_2 \cdot g_2 g_1 \in B'_k$ also according to Lemma 1 $x \in B'_k$. Therefore a following inclusion holds $(g_1 g_2, g_2 g_1) = g^2 \in B'_k$. In first case the proof is even simpler because $g_1^2, g_2^2 \in B'$ by the induction hypothesis.

Lemma 6. *If an element $g = (g_1, g_2) \in G'_k$ then $g_1, g_2 \in G_{k-1}$ and $g_1 g_2 \in B'_{k-1}$.*

Proof. As $B'_k < G_k$ therefore it is enough to show that $g_1 \in G_{k-1}$ and $g_1 g_2 \in B'_{k-1}$. Let us fix some $g = (g_1, g_2) \in G'_k < B'_k$. Then Lemma 1 implies that $g_1 g_2 \in B'_{k-1}$.

In order to show that $g_1 \in G_{k-1}$ we firstly consider just one commutator of arbitrary elements from G_k

$$f = (f_1, f_2)\sigma, \ h = (h_1, h_2)\pi \in G_k,$$

where $f_1, f_2, h_1, h_2 \in B_{k-1}$, $\sigma, \pi \in C_2$. The definition of G_k implies that $f_1 f_2, h_1 h_2 \in G_{k-1}$.

If $g = (g_1, g_2) = [f, h]$ then

$$g_1 = f_1 h_i f_j^{-1} h_k^{-1}$$

for some $i, j, k \in \{1, 2\}$. Then

$$g_1 = f_1 h_i f_j (f_j^{-1})^2 h_k (h_k^{-1})^2 = (f_1 f_j)(h_i h_k) x (f_j^{-1} h_k^{-1})^2,$$

where x is product of commutators of f_i, h_j and f_i, h_k, hence $x \in B'_{k-1}$.

It is enough to consider first product $f_1 f_j$. If $j = 1$ then $f_1^2 \in B'_{k-1}$ by Proposition 2 if $j = 2$ then $f_1 f_2 \in G_{k-1}$ according to definition of G_k, the same is true for $h_i h_k$. Thus, for any i, j, k it holds $f_1 f_j, h_i h_k \in G_{k-1}$. Besides that a square $(f_j^{-1}h_k^{-1})^2 \in B_k'$ according to Proposition 2. Therefore $g_1 \in G_{k-1}$ because of Proposition 2 and Proposition 1, the same is true for g_2.

Now it lefts to consider the product of some $f = (f_1, f_2), h = (h_1, h_2)$, where $f_1, h_1 \in G_{k-1}$, $f_1 h_1 \in G_{k-1}$ and $f_1 f_2, h_1 h_2 \in B_{k-1}'$

$$fh = (f_1 h_1, f_2 h_2).$$

Since $f_1 f_2, h_1 h_2 \in B_{k-1}'$ by imposed condition in this item and taking into account that $f_1 h_1 f_2 h_2 = f_1 f_2 h_1 h_2 x$ for some $x \in B_{k-1}'$ then $f_1 h_1 f_2 h_2 \in B_{k-1}'$ by Lemma 1. Other words closedness by multiplication holds and so according Lemma1 we have element of commutator G_k'.

In the following theorem we prove 2 facts at once.

Theorem 4. *The following statements are true.*

1. *An element $g = (g_1, g_2) \in G_k'$ iff $g_1, g_2 \in G_{k-1}$ and $g_1 g_2 \in B_{k-1}'$.*
2. *Commutator subgroup G_k' coincides with set of all commutators for $k \geq 1$*

$$G_k' = \{[f_1, f_2] \mid f_1 \in G_k, f_2 \in G_k\}.$$

Proof. For the case $k = 1$ we have $G_1' = \langle e \rangle$. So, further we consider the case $k \geq 2$.

Sufficiency of the first statement of this theorem follows from the Lemma 6. So, in order to prove necessity of the both statements it is enough to show that element

$$w = (r_1, r_1^{-1} x),$$

where $r_1 \in G_{k-1}$ and $x \in B_{k-1}'$, can be represented as a commutator of elements from G_k. By Proposition 3 we have $x = [f, g]$ for some $f \in B_{k-1}$ and $g \in G_{k-1}$. Therefore

$$w = (r_1, r_1^{-1}[f, g]).$$

By the Corollary 1 we can represent w as a commutator of

$$(e, a_{1,2})\sigma \in B_k \text{ and } (a_{2,1}, a_{2,2}) \in B_k,$$

where $a_{2,1} = (f^{-1})^{r_1^{-1}}, a_{2,2} = r_1 a_{2,1}, a_{1,2} = g^{a_{2,2}^{-1}}$. It only lefts to show that $(e, a_{1,2})\sigma$, $(a_{2,1}, a_{2,2}) \in G_k$. Note the following

$a_{1,2} = g^{a_{2,2}^{-1}} \in G_{k-1}$ by Corollary 6.

$a_{2,1} a_{2,2} = a_{2,1} r_1 a_{2,1} = r_1 [r_1, a_{2,1}] a_{2,1}^2 \in G_{k-1}$ by Proposition 1 and Proposition 2.

So we have $(e, a_{1,2})\sigma \in G_k$ and $(a_{2,1}, a_{2,2}) \in G_k$ by the definition of G_k.

Proposition 3. *For arbitrary $g \in G_k$ the inclusion $g^2 \in G_k'$ holds.*

Proof. Induction on k: elements of G_1^2 have form $(\sigma)^2 = e$, where $\sigma = (1, 2)$, so the statement holds. In general case, when $k > 1$, the elements of G_k have the form $g = (g_1, g_2)\sigma^i$, $g_1, g_2 \in B_{k-1}$, $i \in \{0, 1\}$. Then we have two possibilities: $g^2 = (g_1^2, g_2^2)$ or $g^2 = (g_1 g_2, g_2 g_1)$.

Firstly we show that $g_1^2 \in G_{k-1}, g_2^2 \in G_{k-1}$. According to Proposition 2, we have $g_1^2, g_2^2 \in B_{k-1}'$ and according to Proposition 1, we have $B_{k-1}' < G_{k-1}$ then using Theorem 4 $g^2 = (g_1^2, g_2^2) \in G_k$.

Consider the second case $g^2 = (g_1 g_2, g_2 g_1)$. Since $g \in G_k$, then, according to the definition of G_k we have that $g_1 g_2 \in G_{k-1}$. By Proposition 1, and definition of G_k, we obtain

$$g_2 g_1 = g_1 g_2 g_2^{-1} g_1^{-1} g_2 g_1 = g_1 g_2 [g_2^{-1}, g_1^{-1}] \in G_{k-1},$$
$$g_1 g_2 \cdot g_2 g_1 = g_1 g_2^2 g_1 = g_1^2 g_2^2 [g_2^{-2}, g_1^{-1}] \in B_{k-1}'.$$

Note that $g_1^2, g_2^2 \in B_{k-1}'$ according to Proposition 2, then $g_1^2 g_2^2 [g_2^{-2}, g_1^{-1}] \in B_{k-1}'$. Since $g_1 g_2 \cdot g_2 g_1 \in B_{k-1}'$ and $g_1 g_2, g_2 g_1 \in G_{k-1}$, then, according to Lemma 6, we obtain $g^2 = (g_1 g_2, g_2 g_1) \in G_k'$.

Statement 1. *The commutator subgroup is a subgroup of G_k^2 i.e. $G'{}_k < G_k^2$.*

Proof. Indeed, an arbitrary commutator presented as product of squares. Let $a,\ b \in G$ and set that $x = a,\ y = a^{-1}ba,\ z = a^{-1}b^{-1}$. Then $x^2y^2z^2 = a^2(a^{-1}ba)^2(a^{-1}b^{-1})^2 = aba^{-1}b^{-1}$, in more detail: $a^2(a^{-1}ba)^2(a^{-1}b^{-1})^2 = a^2a^{-1}ba\,a^{-1}ba\ a^{-1}b^{-1}a^{-1}b^{-1} =$
$= abbb^{-1}a^{-1}b^{-1} = [a, b]$. In such way we obtain all commutators and their products. Thus, we generate by squares the whole $G'{}_k$.

Corollary 7. *For the Syllow subgroup $(Syl_2 A_{2^k})$ the following equalities $Syl_2' A_{2^k} = (Syl_2 A_{2^k})^2$, $\Phi(Syl_2 A_{2^k}) = Syl_2' A_{2^k}$, that are characteristic properties of special p-groups [22], are true.*

Proof. As well known, for an arbitrary group (also by Statement 1) the following embedding $G' \triangleleft G^2$ holds. In view of the above Proposition 3, a reverse embedding for G_k is true. Thus, the group $Syl_2 A_{2^k}$ has some properties of special p-groups that is $P' = \Phi(P)$ [22] because $G_k^2 = G_k'$ and so $\Phi(Syl_2 A_{2^k}) = Syl_2'(A_{2^k})$.

Corollary 8. *Commutator width of the group $Syl_2 A_{2^k}$ equals to 1 for $k \geq 2$.*

It immediately follows from item 2 of Theorem 4.

3. Minimal generating set

For the construction of minimal generating set we used the representation of elements of group G_k by portraits of automorphisms at restricted binary tree $AutX^k$. For convenience we will identify elements of G_k with its faithful representation by portraits of automorphisms from $AutX^{[k]}$.

We denote by $A|_l$ a set of all functions a_l, such, that $[\varepsilon, \ldots, \varepsilon, a_l, \varepsilon, \ldots] \in [A]_l$. Recall that, according to [21], l-coordinate subgroup $U < G$ is the following subgroup.

Definition 1. *For an arbitrarry $k \in \mathbb{N}$ we call a $k-$*coordinate *subgroup $U < G$ a subgroup, which is determined by k-coordinate sets $[U]_l$, $l \in \mathbb{N}$, if this subgroup consists of all Kaloujnine's tableaux $a \in I$ for which $[a]_l \in [U]_l$.*

We denote by $G_k(l)$ a level subgroup of G_k, which consists of the tuples of v.p. from X^l, $l < k-1$ of any $\alpha \in G_k$. We denote as $G_k(k-1)$ such subgroup of G_k that is generated by v.p., which are located on X^{k-1} and isomorphic to W_{k-1}. Note that $G_k(l)$ is in bijective correspondence (and isomorphism) with l-coordinate subgroup $[U]_l$ [21].

For any v.p. g_{li} in v_{li} of X^l we set in correspondence with g_{li} the permutation $\varphi(g_{li}) \in S_2$ by the following rule:

$$\varphi(g_{li}) = \begin{cases} (1,2), & \text{if } g_{li} \neq e, \\ e, & \text{if } g_{li} = e. \end{cases} \tag{7}$$

Define a homomorphic map from $G_k(l)$ onto S_2 with the kernel consisting of all products of even number of transpositions that belongs to $G_k(l)$. For instance, the element (12)(34) of $G_k(2)$ belongs to $ker\varphi$. Hence, $\varphi(g_{li}) \in S_2$.

Definition 2. *We define the subgroup of l-th level as a subgroup generated by all possible vertex permutation of this level.*

Statement 2. *In ${G_k}'$, the following k equalities are true:*

$$\prod_{l=1}^{2^l} \varphi(g_{lj}) = e, \quad 0 \leq l < k-1. \tag{8}$$

For the case $i = k-1$, the following condition holds:

$$\prod_{j=1}^{2^{k-2}} \varphi(g_{k-1j}) = \prod_{j=2^{k-2}+1}^{2^{k-1}} \varphi(g_{k-1j}) = e. \tag{9}$$

Thus, ${G'}_k$ has k new conditions on a combination of level subgroup elements, except for the condition of last level parity from the original group.

Proof. Note that the condition (8) is compatible with that were founded by R. Guralnik in [23], because as it was proved by author [16] $G_{k-1} \simeq B_{k-2} \rtimes \mathcal{W}_{k-1}$, where $B_{k-2} \simeq \underset{i=1}{\overset{k-2}{\wr}} C_2^{(i)}$.

According to Property 1, $G'_k \leq G_k^2$, so it is enough to prove the statement for the elements of G_k^2. Such elements, as it was described above, can be presented in the form $s = (s_{l1}, ..., s_{l2^l})\sigma$, where $\sigma \in G_{l-1}$ and s_{li} are states of $s \in G_k$ in v_{li}, $i \leq 2^l$. For convenience we will make the transition from the tuple $(s_{l1}, ..., s_{l2^l})$ to the tuple $(g_{l1}, ..., g_{l2^l})$. Note that there is the trivial vertex permutation $g_{lj}^2 = e$ in the product of the states $s_{lj} \cdot s_{lj}$.

Since in G'_k v.p. on X^0 are trivial, so σ can be decomposed as $\sigma = (\sigma_{11}, \sigma_{21})$, where σ_{21}, σ_{22} are root permutations in v_{11} and v_{12}.

Consider the square of s. So we calculate squares $((s_{l1}, s_{l2}, ..., s_{l2^{l-1}})\, \sigma)^2$. The condition (8) is equivalent to the condition that s^2 has even index on each level. Two cases are feasible: if permutation $\sigma = e$, then $((s_{l1}, s_{l2}, ..., s_{l2^{l-1}})\, \sigma)^2 = \left(s_{l1}^2, s_{l2}^2, ..., s_{l2^{l-1}}^2\right) e$, so after the transition from $\left(s_{l1}^2, s_{l2}^2, ..., s_{l2^{l-1}}^2\right)$ to $\left(g_{l1}^2, g_{l2}^2, ..., g_{l2^{l-1}}^2\right)$, we get a tuple of trivial permutations $(e,\ ...\ , e)$ on X^l, because $g_{lj}^2 = e$. In general case, if $\sigma \neq e$, after such transition we obtain $\left(g_{l1} g_{l\sigma(2)},\ \dots\ ,\ g_{l2^{l-1}} g_{l\sigma(2^{l-1})}\right) \sigma^2$. Consider the product of form

$$\prod_{j=1}^{2^l} \varphi(g_{lj} g_{l\sigma(j)}), \tag{10}$$

where σ and $g_{li} g_{l\sigma(i)}$ are from $\left(g_{l1} g_{l\sigma(2)},\ \dots\ ,\ g_{l2^{l-1}} g_{l\sigma(2^{l-1})}\right) \sigma^2$.

Note that each element g_{lj} occurs twice in (10) regardless of the permutation σ, therefore considering commutativity of homomorphic images $\varphi(g_{lj})$, $1 \leq j \leq 2^l$ we conclude that $\prod\limits_{j=1}^{2^l} \varphi(g_{lj} g_{l\sigma(j)}) = \prod\limits_{j=1}^{2^l} \varphi(g_{lj}^2) = e$, because of $g_{lj}^2 = e$. We rewrite $\prod\limits_{j=1}^{2^l} \varphi(g_{lj}^2) = e$ as characteristic condition: $\prod\limits_{j=1}^{2^{l-1}} \varphi(g_{lj}) = \prod\limits_{j=2^{l-1}+1}^{2^l} \varphi(g_{lj}) = e$.

According to Property 1, any commutator from G'_k can be presented as a product of some squares s^2, $s \in G_k$, $s = ((s_{l1}, ..., s_{l2^l})\sigma\,)$.

A product of elements of $G_k(k-1)$ satisfies the equation $\prod\limits_{j=1}^{2^l} \varphi(g_{lj}) = e$, because any permutation of elements from X^k, which belongs to G_k is even. Consider the element $s = (s_{k-1,1}, ..., s_{k-1,2^{k-1}})\sigma$, where $(s_{k-1,1}, ..., s_{k-1,2^{k-1}}) \in$

$G_k(k-1)$, $\sigma \in G_{k-1}$. If $g_{01} = (1,2)$, where g_{01} is root permutation of σ, then $s^2 = (s_{k-1,1}s_{k-1\sigma(1)}, ..., s_{k-1,(2^{k-1})}s_{k-1,\sigma(2^{k-1})})$, where $\sigma(j) > 2^{k-1}$ for $j \le 2^{k-1}$, and if $j < 2^{k-1}$ then $\sigma(j) \ge 2^{k-1}$. Because of $\prod_{j=1}^{2^{k-1}} \varphi(g_{k-1,j}) = e$ in G_k and the property $\sigma(j) \le 2^{k-1}$ for $j > 2^{k-1}$, then the product $\prod_{j=1}^{2^{k-2}} \varphi(g_{k-1,j}g_{k-1,\sigma(j)})$ of images of v.p. from $(g_{k-1,1}g_{k-1,\sigma(1)}, ..., g_{k-1,(2^{k-1})}g_{k-1,\sigma(2^{k-1})})$ is equal to $\prod_{j=1}^{2^{k-1}} \varphi(g_{k-1,j}) = e$. Indeed in $\prod_{j=1}^{2^{k-1}} \varphi(g_{k-1,j})$ and as in $\prod_{j=1}^{2^{k-1}} \varphi(g_{k-1,j}g_{k-1,\sigma(j)})$ are the same v.p. from X^{k-1} regardless of such σ as described above.

The same is true for right half of X^{k-1}. Therefore the equality (9) holds.

Note that such product $\prod_{j=1}^{2^{k-1}} \varphi(g_{k-1,j})$ is homomorphic image of $(g_{l,1}g_{l,\sigma(1)}, ..., g_{l,(2^l)}g_{l\sigma(2^l)})$, where $l = k-1$, as an element of $G'_k(l)$ after mapping (7).

If $g_{01} = e$, where g_{01} is root permutation of σ then σ can be decomposed as $\sigma = (\sigma_{11}, \sigma_{12})$, where σ_{11}, σ_{12} are root permutations in v_{11} and v_{12}. As a result s^2 has a form $\left((s_{l1}s_{l\sigma(1)}, ..., s_{l\sigma(2^{l-1})})\sigma_1^2, (s_{l2^{l-1}+1}s_{l\sigma(2^{l-1}+1)}, ..., s_{l(2^l)}s_{l\sigma(2^l)})\sigma_2^2\right)$, where $l = k-1$. As a result of action of σ_{11} all states of l-th level with number $1 \le j \le 2^{k-2}$ permutes in coordinate from 1 to 2^{k-2} the other are fixed. The action of σ_{11} is analogous.

It corresponds to the next form of element from $G'_k(l)$:

$$(g_{l1}g_{l\sigma_1(1)}, ..., g_{l\sigma_1(2^{l-1})}), \quad (g_{l2^{l-1}+1}g_{l\sigma_2(2^{l-1}+1)}, ..., g_{l(2^l)}g_{l\sigma_2(2^l)}).$$

Therefore the product of form

$$\prod_{j=1}^{2^{k-2}} \varphi(g_{k-1,j}g_{l\sigma(j)}) = \prod_{j=2^{k-2}+1}^{2^{k-1}} \varphi(g_{k-1,j}^2) = e,$$

because of $g_{k-1,j}^2 = e$. Thus, characteristic equation (9) of $k-1$ level holds.

The conditions (8), (9) for every s^2, $s \in G_k$ hold so it holds for their product that is equivalent to conditions hold for every commutator.

Definition 3. *We define a subdirect product of group G_{k-1} with itself by equipping it with condition (8) and (9) of index parity on all of $k-1$ levels.*

Corollary 9. *The subdirect product $G_{k-1} \boxtimes G_{k-1}$ is defined by $k-2$ outer relations on level subgroups. The order of $G_{k-1} \boxtimes G_{k-1}$ is 2^{2^k-k-2}.*

Proof. We specify a subdirect product for the group $G_{k-1} \boxtimes G_{k-1}$ by using $(k-2)$ conditions for the subgroup levels. Each G_{k-1} has even index on $k-2$-th level, it implies that its relation for $l=k-1$ holds automatically. This occurs because of the conditions of parity for the index of the last level is characteristic of each of the multipliers G_{k-1}. Therefore It is not an essential condition for determining a subdirect product.

Thus, to specify a subdirect product in the group $G_{k-1} \boxtimes G_{k-1}$, there are obvious only $k-2$ outer conditions on subgroups of levels. Any of such conditions reduces the order of $G_{k-1} \times G_{k-1}$ in 2 times. Hence, taking into account that the order of G_{k-1} is $2^{2^{k-1}-2}$, the order of $G_{k-1} \boxtimes G_{k-1}$ as a subgroup of $G_{k-1} \times G_{k-1}$ the following: $|G_{k-1} \boxtimes G_{k-1}| = \left(2^{2^{k-1}-2}\right)^2 : 2^{k-2} = 2^{2^k-4} : 2^{k-2} = 2^{2^k-k-2}$. Thus, we use $k-2$ additional conditions on level subgroup to define the subdirect product $G_{k-1} \boxtimes G_{k-1}$, which contain G'_k as a proper subgroup of G_k. Because according to the conditions, which are realized in the commutator of G'_k, (9) and (8) indexes of levels are even.

Corollary 10. *A commutator G'_k is embedded as a normal subgroup in $G_{k-1} \boxtimes G_{k-1}$.*

Proof. A proof of injective embedding G'_k into $G_{k-1} \boxtimes G_{k-1}$ immediately follows from last item of proof of Corollary 9. The minimality of G'_k as a normal subgroup of G_k and injective embedding G'_k into $G_{k-1} \boxtimes G_{k-1}$ immediately entails that $G'_k \triangleleft G_{k-1} \boxtimes G_{k-1}$.

Theorem 5. *A commutator of G_k has form $G'_k = G_{k-1} \boxtimes G_{k-1}$, where the subdirect product is defined by relations (8) and (9). The order of G'_k is 2^{2^k-k-2}.*

Proof. Since according to Statement 2 (g_1, g_2) as elements of G'_k also satisfy relations (8) and (9), which define the subdirect product $G_{k-1} \boxtimes G_{k-1}$. Also condition $g_1 g_2 \in B'_{k-1}$ gives parity of permutation which defined by (g_1, g_2) because B'_{k-1} contains only element with even index of level [16]. The group G'_k has 2 disjoint domains of transitivity so G'_k has the structure of a subdirect product of G_{k-1} which acts on this domains transitively. Thus, all elements of G'_k satisfy the conditions (8), (9) which define subdirect product $G_{k-1} \boxtimes G_{k-1}$. Hence $G'_k < G_{k-1} \boxtimes G_{k-1}$ but G'_k can be equipped by some other relations, therefore, the presence of isomorphism has not yet been proved. For proving revers inclusion we have to show that every element from $G_{k-1} \boxtimes G_{k-1}$ can be expressed as word $a^{-1}b^{-1}ab$, where $a, b \in G_k$. Therefore, it suffices to show the reverse inclusion. For this goal we use that $G'_k < G_{k-1} \boxtimes G_{k-1}$. As it was shown in [16] that the order of G_k is 2^{2^k-2}.

As it was shown above, G'_k has k new conditions relatively to G_k. Each condition is stated on some level-subgroup. Each of these conditions reduces an order of the corresponding level subgroup in 2 times, so the order of G'_k is in 2^k times lesser. On every X^l, $l \leq k-1$, there is even number of active v.p. by this reason, there is trivial permutation on X^0.

According to the Corollary 9, in the subdirect product $G_{k-1} \boxtimes G_{k-1}$ there are exactly $k-2$ conditions relatively to $G_{k-1} \times G_{k-1}$, which are for the subgroups of levels. It has been shown that the relations (8), (9) are fulfilled in G'_k.

Let α_{lm}, $0 \leq l \leq k-1$, $0 \leq m \leq 2^{l-1}$ be an automorphism from G_k having only one active v.p. in v_{lm}, and let α_{lm} have trivial permutations in rest of the vertices. Recall that partial case of notation of form α_{lm} is the generator $\alpha_l := \alpha_{l1}$ of G_k which was defined by us in [16] and denoted by us as α_l. Note that the order of α_{li}, $0 \leq l \leq k-1$ is 2. Thus, $\alpha_{ji} = \alpha_{ji}^{-1}$. We choose a generating set consisting of the following $2k-3$ elements: $(\alpha_{1,1;2}), \alpha_{2,1}, ..., \alpha_{k-1,1}, \alpha_{2,3}, ..., \alpha_{k-1,2^{k-2}+1}$, where $(\alpha_{1,1;2})$ is an automorphism having exactly 2 active v.p. in v_{11} and v_{12}. Product of the form $(\alpha_{j1}\alpha_{l1}\alpha_{j1})\alpha_{l1}$ are denoted by P_{lm}. In more details, $P_{lm} = \alpha_{ji}\alpha_{lm}\alpha_{ji}\alpha_{lm}$, where $\alpha_{ji} \in G_k(j)$. Using a conjugation by generator α_j, $0 \leq j < l$ we can express any v.p. on l-level, because $(\alpha_j\alpha_l\alpha_j) = \alpha_{l2^{l-j-1}+1}$. Consider the product $P_{lj} = (\alpha_j\alpha_l\alpha_j)\alpha_l$.

1. We need to show that every element of $G_{k-1} \boxtimes G_{k-1}$ can be constructed as $g^{-1}h^{-1}gh$, $g, h \in G_k$. This proves the absence of other relations in G'_k except those that in the subdirect product $G_{k-1} \boxtimes G_{k-1}$. Thereby we prove the embeddedness of G'_k in $G_{k-1} \boxtimes G_{k-1}$. We have to construct an element of form $P_{k-1}P_{k-2} \cdot ... \cdot P_1P_0$ as a product of elements of form $[g, h]$, where $P_l = \prod_{i=1}^{2^l} P_{lm}$ satisfying relations (8), (9).

2. We have to construct an arbitrary tuple of 2 active v.p. on X^l as a product of several P_l. We use the generator α_l and conjugating it by α_j, $j < l$. It corresponds to the tuple of v.p. of the form $(g_{l1}, e, ..., e, g_{lj}, e, ..., e)$, where g_{l1}, g_{lj} are non-trivial. Note that this tuple $(g_{l1}, e, ..., e, g_{lj}, e, ..., e)$ is an element of direct product if we consider as an element of S_2 in vertices of X^l. To obtain a tuple of v.p. of form $(e, ..., e, g_{lm}, e, ..., e, g_{lj}, e, ..., e)$ we multiply P_{lj} and P_{lm}.

3. To obtain a tuple of v.p. with $2m$ active v.p. we construct $\prod_{i=1}^{m} P_{lj_i}$, $m < 2^l$ for varying $i, j < 2^{k-2}$.

On the $(k-1)$-th level we choose the generator τ which was defined in [16] as $\tau = \tau_{k-1,1}\tau_{k-1,\,2^{k-1}}$. Recall that it was shown in [16] how to express any τ_{ij} using τ, $\tau_{i,2^{k-2}}$, $\tau_{j,2^{k-2}}$, where $i, j < 2^{k-2}$, as a product of commutators $\tau_{ij} = \tau_{i,2^{k-2}}\tau_{j,2^{k-2}} = (\alpha_i^{-1}\tau_{1,2^{k-2}}^{-1}\alpha_i\tau_{j,2^{k-2}})$. Here $\tau_{i,2^{k-2}}$ was expressed as the commutator $\tau_{i,2^{k-2}} = \alpha_i^{-1}\tau_{1,2^{k-2}}^{-1}\alpha_i\tau_{1,2^{k-2}}$. Thus, we express all tuples of elements satisfying to relations (8) and (9) by using only commutators of G_k.

Thus, we get all tuples of each level subgroup elements satisfying the relations (8) and (9). It means we express every element of each level subgroup by a commutators. In particular to obtain a tuple of v.p. with $2m$ active v.p. on X^{k-2} of $v_{11}X^{[k-1]}$, we will construct the product for τ_{ij} for varying $i, j < 2^{k-2}$.

Thus, all vertex labelings of automorphisms, which appear in the representation of $G_{k-1} \boxtimes G_{k-1}$ by portraits as the subgroup of $AutX^{[k]}$, are also in the representation of G'_k.

Since there are faithful representations of $G_{k-1} \boxtimes G_{k-1}$ and G'_k by portraits of automorphisms from $AutX^{[k]}$, which coincide with each other, then subgroup G'_k of $G_{k-1} \boxtimes G_{k-1} \simeq G'_k$ is equal to whole $G_{k-1} \boxtimes G_{k-1}$ (i.e. $G_{k-1} \boxtimes G_{k-1} = G'_k$).

The archived results are confirmed by algebraic system GAP calculations. For instance, $|Syl_2 A_8| = 2^6 = 2^{2^3-2}$ and $|(SylA_{2^3})'| = 2^{2^3-3-2} = 8$. The order of G_2 is 4, the number of additional relations in subdirect product is $k-2 = 3-2 = 1$. Then we have the same result $(4 \cdot 4) : 2^1 = 8$, which confirms Theorem 5.

Example 1. *Set $k = 4$ then $|(SylA_{16})'| = |(G_4)'| = 1024$, $|G_3| = 64$, since $k-2 = 2$, so according to our theorem above order of $Syl_2 A_{16} \boxtimes Syl_2 A_{16}$ is defined by $2^{k-2} = 2^2$ relations, and by this reason is equal to $(64 \cdot 64) : 4 = 1024$. Thus, orders are coincides.*

Example 2. *The true order of $(Syl_2 A_{32})'$ is $33554432 = 2^{25}$, $k = 5$. A number of additional relations which define the subdirect product is $k-2 = 3$. Thus, according to Theorem 5, $| \ (Syl_2 A_{16} \boxtimes Syl_2 A_{16})' \ | = 2^{14} 2^{14} : 2^{5-2} = 2^{28} : 2^{5-2} = 2^{25}$.*

According to calculations in GAP we have: $Syl_2 A_7 \simeq Syl_2 A_6 \simeq D_4$. Therefore its derived subgroup $(Syl_2 A_7)' \simeq (Syl_2 A_6)' \simeq (D_4)' = C_2$.

The following structural law for Syllows 2-subgroups is typical. The structure of $Syl_2 A_n$, $Syl_2 A_k$ is the same. If for all n and k that have the same multiple of 2 as multiplier in decomposition on $n!$ and $k!$ Thus, $Syl_2 A_{2k} \simeq Syl_2 A_{2k+1}$.

Example 3. *$Syl_2 A_7 \simeq Syl_2 A_6 \simeq D_4$, $Syl_2 A_{10} \simeq Syl_2 A_{11} \simeq Syl_2 S_8 \simeq (D_4 \times D_4) \rtimes C_2$. $Syl_2 A_{12} \simeq Syl_2 S_8 \boxtimes Syl_2 S_4$, by the same reasons that from the proof of Corollary 9 its commutator subgroup is decomposed as $(Syl_2 A_{12})' \simeq (Syl_2 S_8)' \times (Syl_2 S_4)'$.*

Lemma 7. *In G''_k the following equalities are true:*

$$\prod_{j=1}^{2^{l-2}} \varphi(g_{lj}) = \prod_{j=2^{l-2}+1}^{2^{l-1}} \varphi(g_{lj}) = \prod_{j=2^{l-1}+1}^{2^{l-1}+2^{l-2}} \varphi(g_{lj}) = \\ = \prod_{j=2^{l-1}+2^{l-2}+1}^{2^{l}} \varphi(g_{lj}), \quad 2 < l < k \tag{11}$$

In case $l = k-1$, the following conditions hold:

$$\prod_{j=1}^{2^{l-2}} \varphi(g_{lj}) = \prod_{j=2^{i-1}+1}^{2^{l-1}} \varphi(g_{lj}) = e, \quad \prod_{j=2^{l-1}}^{2^{l-1}+2^{l-2}} \varphi(g_{lj}) = \prod_{j=2^{l-1}+2^{l-2}}^{2^{l}} \varphi(g_{lj}) = e \tag{12}$$

In other terms, the subgroup G''_k has an even index of any level of $v_{11}X^{[k-2]}$ and of $v_{12}X^{[k-2]}$.

Proof. As a result of derivation of G'_k, elements of $G''_k(1)$ are trivial. Due the fact that $G'_k \simeq G_{k-1} \boxtimes G_{k-1}$, we can derivate G'_k by commponents. The commutator of G_{k-1} is already investigated in Theorem 5. As $G^2_{k-1} = G'_{k-1}$ by Corollary 7, it is more convenient to present a characteristic equalities in the second commutator $G''_k \simeq G'_{k-1} \boxtimes G'_{k-1}$ as equations in $G^2_{k-1} \boxtimes G^2_{k-1}$. As shown above, for $2 \leq l < k-1$, in G^2_{k-1} the following equalities are true:

$$\prod_{j=1}^{2^{l-1}} \varphi(g_{lj} g_{l\sigma(j)}) = \prod_{j=1}^{2^{l-1}} \varphi(g_{lj}) \prod_{j=1}^{2^{l-1}} \varphi(g_{l\sigma(j)}) = \\ = \prod_{j=1}^{2^{l-1}} \varphi(g_{lj}) \prod_{j=1}^{2^{l-1}} \varphi(g_{li}) = \prod_{j=1}^{2^{l-1}} \varphi(g^2_{lj}) = e \tag{13}$$

$$\prod_{j=1}^{2^{l-2}} \varphi(g_{lj}) = \prod_{j=2^{l-2}+1}^{2^{l-1}} \varphi(g_{lj}) = \prod_{j=2^{l-1}+1}^{2^{l-1}+2^{l-2}} \varphi(g_{lj}) = \prod_{j=2^{l-1}+2^{l-2}+1}^{2^{l}} \varphi(g_{lj}). \tag{14}$$

The equality (14) is true because of it is the initial group $G'_k \simeq G_{k-1} \boxtimes G_{k-1}$. The equalities

$$\prod_{j=2^{l-1}+1}^{2^{l-1}+2^{l-2}} \varphi(g_{lj}) = \prod_{j=2^{l-1}+2^{l-2}+1}^{2^{l}} \varphi(g_{lj})$$

are right for elements of second group G'_{k-1}, since elements of the original group are endowed with this conditions.

Upon a squaring of G'_k any element of $G'_k(l)$, satisfies the equality (14) in addition to satisfying the previous conditions (11) because of $(G_{k-1}(l))^2 = G'_{k-1}(l)$. The similar conditions appears in $(G'_{k-1}(k-2))^2$ after squaring of G'_k. Thus, taking into account the characteristic equations of $G'_{k-1}(l)$, the subgroup $(G'_{k-1}(k-2))^2$ satisfies the equality:

$$\prod_{j=1}^{2^{k-3}} \varphi(g_{lj}) = \prod_{j=2^{k-3}+1}^{2^{k-2}} \varphi(g_{lj}) = e, \quad \prod_{j=2^{k-2}+1}^{2^{k-2}+2^{k-3}} \varphi(g_{lj}) = \prod_{j=2^{k-1}+2^{k-2}+1}^{2^{k-1}} \varphi(g_{lj}) = e. \tag{15}$$

Taking into account the structure $G'_k \simeq G_{k-1} \boxtimes G_{k-1}$ we obtain after derivation $G''_k \simeq (G_{k-2} \boxtimes G_{k-2}) \boxtimes (G_{k-2} \boxtimes G_{k-2})$. With respect to conditions 8, 9 in the subdirect product we have that the order of G''_k is $2^{2^k-k-2} : 2^{2k-3} = 2^{2^k-3k+1}$ because on every level $2 \leq l < k$ order of level subgroup $G''_k(l)$ is in 4 times lesser then order of $G'_k(l)$. On the 1-st level one new condition arises that reduce order of $G'_k(1)$ in 2 times. Totally we have $2(k-2)+1 = 2k-3$ new conditions in comparing with G'_k.

Example 4. *Size of (G''_4) is 32, a size of direct product $(G'_3)^2$ is 64, but, due to relation on second level of G''_k, the direct product $(G'_3)^2$ transforms into the subdirect product $G'_3 \boxtimes G'_3$ that has 2 times less feasible combination on X^2. The number of additional relations in the subdirect product is $k-3 = 4-3 = 1$. Thus the order of product is reduced in 2^1 times.*

Example 5. *The commutator subgroup of $Syl'_2(A_8)$ consists of elements:*

$$\{e, (13)(24)(57)(68), (12)(34), (14)(23)(57)(68), (56)(78), \\ (13)(24)(58)(67), (12)(34)(56)(78), (14)(23)(58)(67)\}.$$

The commutator $Syl'_2(A_8) \simeq C_2^3$ that is an elementary abelian 2-group of order 8. This fact confirms our formula $d(G_k) = 2k-3$, because $k = 3$ and $d(G_k) = 2k-3 = 3$. A minimal generating set of $Syl'_2(A_8)$ consists of 3 generators: $(1,3)(2,4)(5,7)(6,8)$, $(1,2)(3,4)$, $(1,3)(2,4)(5,8)(6,7)$.

Example 6. *The minimal generating set of $Syl'_2(A_{16})$ consists of 5 (that is $2 \cdot 4 - 3$) generators:*

$$(1,4,2,3)(5,6)(9,12)(10,11), (1,4)(2,3)(5,8)(6,7), (1,2)(5,6), \\ (1,7,3,5)(2,8,4,6)(9,14,12,16)(10,13,11,15), \\ (1,7)(2,8)(3,6)(4,5)(9,16,10,15)(11,14,12,13).$$

Example 7. *Minimal generating set of $Syl_2'(A_{32})$ consists of 7 (that is $2\cdot 5-3$) generators:*

$$(23,24)(31,32), (1,7)(2,8)(3,5,4,6)(11,12)(25,32)(26,31)(27,29)(28,30),$$
$$(3,4)(5,8)(6,7)(13,14)(23,24)(27,28)(29,32)(30,31),$$
$$(7,8)(15,16)(23,24)(31,32),$$
$$(1,9,7,15)(2,10,8,16)(3,11,5,13)(4,12,6,14)(17,29,22,27,18,30,21,28)$$
$$(19,32,23,26,20,31,24,25), (1,5,2,6)(3,7,4,8)(9,15)(10,16)(11,13)\times$$
$$(12,14)(19,20)(21,24,22,23)(29,31)(30,32), (3,4)(5,8)(6,7)(9,11,10,12)\times$$
$$(13,14)(15,16)(17,23,20,22,18,24,19,21)(25,29,27,32,26,30,28,31).$$

This confirms our formula of minimal generating set size $2\cdot k-3$.

4. Conclusion

The size of minimal generating set for commutator of Sylow 2-subgroup of alternating group A_{2^k} was proven is equal to $2k-3$.

A new approach to presentation of Sylow 2-subgroups of alternating group A_{2^k} was applied. As a result the short proof of a fact that commutator width of Sylow 2-subgroups of alternating group A_{2^k}, permutation group S_{2^k} and Sylow p-subgroups of $Syl_2 A_{p^k}$ $(Syl_2 S_{p^k})$ are equal to 1 was obtained. Commutator width of permutational wreath product $B \wr C_n$ were investigated.

Скуратовсъкий Р. В.
Мінімальна система твірних комутанта силовських 2-підгруп знакозмінної групи і їх структура

Резюме

Знайдено мінімальну систему твірних для комутанта силовських 2-підгруп знакозмінної групи. Досліджено структуру комутанта силовських 2-підгруп знакозмінної групи A_{2^k}.

Показано, що $(Syl_2 A_{2^k})^2 = Syl'_2 A_{2^k}$, $k > 2$.

Доведено, що довжина по комутатора довільного елемента ітерірованого вінцевого добутку циклічних груп C_{p_i}, $p_i \in \mathbb{N}$ дорівнює 1. Знайдена ширину по коммутанту прямої границі вінцевого добутку циклічних груп. У даній статті знайдені верхні оцінки ширини по комутанту $(cw(G))$ [1] вінцевого добутку груп.

Розглянуто рекурсивне представлення силовских 2-Підгрупп $Syl_2(A_{2^k})$ з A_{2^k}. В результаті отримано коротке доведення того, що ширина по комутанту силовських 2-підгруп груп A_{2^k} і S_{2^k} рівна 1.

Досліджено комутаторна ширина перестановочного сплетення $B \wr C_n$. Знайдена верхня оцінка ширини по коммутанта сплетення груп діючих перестановками — $B \wr C_n$ для довільної групи B.
Ключові слова: вінцевий добуток, мінімальна система твірних комутанта силовських 2-підгруп знакозмінної групи, ширина по комутанту силовських p-підгруп, комутант силовських 2-підгруп знакозмінної групи.

Скуратовский Р. В.
Минимальная система образующих коммутанта силовских 2-подгрупп знакопеременной группы и их структура

Резюме

Найдено минимальная система образующих для коммутанта силовских 2-подгрупп знакопеременной группы. Исследована структура коммутаторной подгруппы силовских 2-подгрупп знакопеременной группы A_{2^k}.

Показано, что $(Syl_2 A_{2^k})^2 = Syl'_2 A_{2^k}$, $k > 2$.

Доказано, что длина по коммутатора произвольного элемента итерированого сплетения циклических групп C_{p_i}, $p_i \in \mathbb{N}$ равна 1. Найдена ширина по коммутанту прямого предела сплетения циклических групп. В данной статье представлены верхние оценки ширины коммутатора $(cw(G))$ [1] сплетения групп.

Рассмотрено рекурсивное представление силовских 2-подгрупп $Syl_2(A_{2^k})$ из A_{2^k}. В результате получено краткое доказательство того, что ширина коммутатора силовских 2-подгрупп группы A_{2^k}, группы перестановок S_{2^k}.

Исследована коммутаторная ширина перестановочного сплетения $B \wr C_n$. Найдена верхняя оценка ширины по коммутанту сплетения групп действующих перестановками — $B \wr C_n$ для произвольной группы B.
Ключевые слова: сплетение групп, минимальная система образующих коммутанта силовских 2-подгрупп знакопеременной группы, ширина по коммутанту силовских p-подгрупп, коммутант силовских 2-подгрупп знакопеременной группы.

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