UDC 539.3

A. A. Fesenko
Odessa I. I. Mechnikov National University, Faculty of Mathematics, Physics and Information Technologies

AN EXACT SOLUTION OF THE DYNAMICAL PROBLEM FOR THE INFINITE ELASTIC LAYER WITH A CYLINDRICAL CAVITY

The wave field of an infinite elastic layer weakened by a cylindrical cavity is constructed in this paper. The ideal contact conditions are given on the upper and bottom faces of the layer. The normal dynamic tensile load is applied to a cylindrical cavity's surface at the initial moment of time. The Laplace and finite $sin-$ and $cos-$ Fourier integral transforms are applied successively directly to axisymmetric equations of motion and to the boundary conditions, on the contrary to the traditional approaches, when integral transforms are applied to solutions' representation through harmonic and biharmonic functions. This operation leads to a one-dimensional vector homogeneous boundary value problem with respect to unknown transformations of displacements. The problem is solved using matrix differential calculus. The field of initial displacements is derived after application of inverse integral transforms. The case of the steady-state oscillations was investigated. The normal stress on the faces of the elastic layer are constructed and investigated depending on the mechanical and dynamic parameters.
MSC: 74B05, 74H05, 74J20.
Key words: exact solution, elastic layer, dynamic load, cylindrical cavity, integral transform.
DOI: 10.18524/2519-206x.2019.2(34).190054.

1. Introduction

The presence of defects in elastic bodies causes a stress concentration and significantly affects at the stress state of constructions. A typical and sufficiently investigated problem of this class is the axisymmetric elasticity problem on the stress state of a layer, weakened by a cylindrical defect, when different boundary conditions are set on layer's faces and defect's surface. Existing research can be divided into three approaches: 1) a construction of an analytic solution of the problem in an explicit form [10], [2]; 2) a construction of an analytical-numerical solution, when the problem is reduced either to an integral equation or to an infinite system of algebraic equations [3], [4]; 3) a numerical solving of the problem [5], [6].

For realization of the first approach, it is essential to satisfy the conditions of ideal contact on a cylindrical surface, when the normal displacements and tangential stress are equal to zero. The exact solution of the formulated problem for the case, when the layer is replaced by a half-space and the stresses are given on the faces, is derived in [5]. An approximate analytical - numerical

Received 17.09.2019

solutions for other boundary conditions on the defect's surface were obtained at the papers [6], [8].

Dynamic statement of the mentioned problem was considered at the papers [9], [10]. The theory of harmonic oscillations and wave propagation in elastic bodies was widely investigated in the monograph [11]. The papers [12], [13] are devoted to the propagation of elastic waves in plates weakened by the cavities or holes. Based on complex function theory, an analytical solution for the dynamic stress concentration due to an arbitrary cylindrical cavity in an infinite inhomogeneous medium was investigated in [14]. The existence of trapped elastic waves above a circular cylindrical cavity in a half-space was demonstrated in [15].

It should be noted that dynamical problems weakened by the defects have found wide application in the practical problems [16], [17]. An experimental method was proposed to explore dynamic failure process of pre-stressed rock specimen with a circular hole to investigate deep underground rock failure at the [18]. The paper [19] proposes a set of exact solutions for three-dimensional dynamic responses of a cylindrical lined tunnel in saturated soil due to internal blast loading are derived by using Fourier transform and Laplace transform. The surrounding soil was modeled as a saturated medium on the basis of Biot's theory and the lining structure modeled as an elastic medium. By utilizing a reliable and efficient numerical method of inverse Laplace transform and Fourier transform, the numerical solutions for the dynamic response of the lining and surrounding soil were obtained.

Nevertheless, the study of an elastic layer hasn't been completed yet and opens up many problems. The main difficulty during the solving of the dynamic problems by the method of integral transforms remains the inversion problem of the Laplace transform. Therefore, it is often necessary to proceed to a more narrow class of the problems about steady state oscillations. Research contributions over the past 50 years on the theory and analysis of elastodynamics are reviewed in the paper [20]. Major topics reviewed are: general theories, steady-state waves in waveguides, transient waves in layered media, diffraction and scattering, and one and two-dimensional theories of elastic bodies. A brief discussion on the direct and inverse problems of elastic waves completes this review.

The problem of elasticity for an infinite layer with a cylindrical cavity in a static statement was considered by G. Ya. Popov [10], where an exact solution was obtained. In this paper this method was extended on the analogical problem in the dynamic statement.

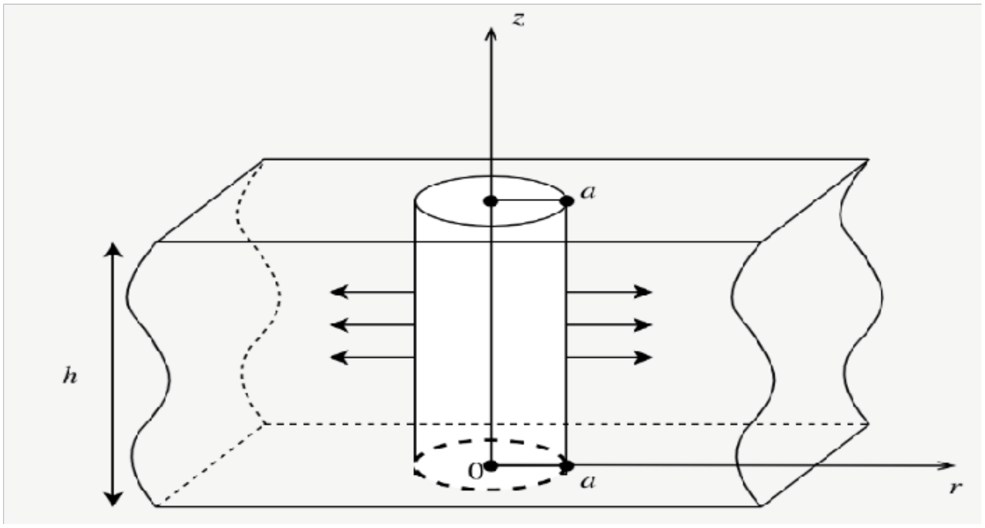

Fig. 1. Geometry of the problem

2. Main Results

1. Statement of the problem. An elastic layer of thickness b (G is a shear modulus, μ is a Poisson's ratio, ρ is density), describing in the cylindrical coordinate system by the correspondences: $a < r < \infty$, $-\pi < \varphi \leq \pi$, $0 \leq z \leq h$ is weakened by a cylindrical cavity $0 \leq r \leq a$, $0 < \varphi \leq \pi$, $0 \leq z \leq b$ (Fig. 1). The layer's upper and bottom faces are in the conditions of ideal contact with a rigid base (the layer is supported by a smooth foundation without a friction)

$$u_r(r,0,t) = 0,\ \tau_{zr}(r,0,t) = 0,\ u_z(r,b,t) = 0,\ \tau_{zr}(r,b,t) = 0 \tag{1}$$

The cylindrical cavity's surface $r = a$ is under the influence of the normal dynamic tensile force $P = p(z,t)$, applied at the initial moment $t = 0$, the tangential loading is absent

$$\sigma_r(a,z,t) = P(z,t),\ \tau_{rz}(a,z,t) = 0 \tag{2}$$

Thus, the problem was reduced to solving axisymmetric equations of motion with respect to the functions $u_r(r,z,t) = u(r,z,t),\ u_z(r,z,t) = w(r,z,t)$ in a cylindrical coordinate system [21]

$$\begin{gathered} r^{-1}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}u(r,z,t)\right] - r^{-2}u(r,z,t) + \frac{\kappa-1}{\kappa+1}\frac{\partial^2}{\partial z^2}u(r,z,t) + \frac{2}{\kappa+1}\frac{\partial^2}{\partial r\partial z}w(r,z,t) = \\ = \frac{\kappa-1}{\kappa+1}\frac{\rho}{G}\frac{\partial^2 u(r,z,t)}{\partial t^2} \\ r^{-1}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}w(r,z,t)\right] + \frac{\kappa+1}{\kappa-1}\frac{\partial^2}{\partial z^2}w(r,z,t) + \frac{2}{\kappa-1}r^{-1}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial z}u(r,z,t)\right] = \\ = \frac{\rho}{G}\frac{\partial^2 w(r,z,t)}{\partial t^2} \end{gathered} \tag{3}$$

where $\kappa = 3 - 4\mu$ and subjected to the mixed boundary conditions (1), (2). Here $c_1^2 = \frac{\kappa+1}{\kappa-1}\frac{G}{\rho}$ - squared velocity of longitudinal wave propagation, $c^2 = \frac{G}{\rho}$ - squared velocity of shear wave propagation. So, $c_1^2 = \frac{\kappa+1}{\kappa-1}c^2$.

The following change of the variables was done

$$\begin{gathered}\rho = a^{-1}r,\ \xi = b^{-1}z,\ \tau = ca^{-1}t,\ u(a\rho, b\xi, ca^{-1}\tau) = U(\rho,\xi,\tau),\\ w(a\rho, b\xi, ca^{-1}\tau) = W(\rho,\xi,\tau)\end{gathered} \tag{4}$$

Consequently, the movement equations (3) can be written in the form

$$\begin{gathered}\rho^{-1}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}U(\rho,\xi,\tau)\right] - \rho^{-2}U(\rho,\xi,\tau) + \frac{\kappa-1}{\kappa+1}\alpha^2\frac{\partial^2}{\partial\xi^2}U(\rho,\xi,\tau)+\\ +\frac{2}{\kappa+1}\alpha\frac{\partial^2}{\partial\rho\partial\xi}W(\rho,\xi,\tau) = \frac{\kappa-1}{\kappa+1}\frac{\partial^2 U(\rho,\xi,\tau)}{\partial\tau^2}\\ \rho^{-1}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}W(\rho,\xi,\tau)\right] + \frac{\kappa+1}{\kappa-1}\alpha^2\frac{\partial^2}{\partial\xi^2}W(\rho,\xi,\tau)+\\ +\rho^{-1}\frac{2}{\kappa-1}\alpha\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\xi}U(\rho,\xi,\tau)\right] = \frac{\partial^2 W(\rho,\xi,\tau)}{\partial t^2}\\ 1<\rho<\infty,\ 0<\xi<1,\ \alpha = \frac{a}{h}.\end{gathered} \tag{5}$$

Boundary conditions (1), taking into account the replacement (4), are transformed into form

$$\frac{\partial}{\partial\xi}U(\rho,0,\tau) = 0,\ \frac{\partial}{\partial\xi}U(\rho,1,\tau) = 0,\ W(\rho,0,\tau) = 0,\ W(\rho,1,\tau) = 0 \tag{6}$$

as the boundary conditions (2) take the form

$$\frac{\partial}{\partial\rho}U(1,\xi,\tau) + \frac{3-\kappa}{1+\kappa}\left[U(1,\xi,\tau) + \alpha\frac{\partial}{\partial\xi}W(1,\xi,\tau)\right] = aG^{-1}\frac{\kappa-1}{\kappa+1}P(\xi,\tau) \tag{7}$$

$$\alpha\frac{\partial}{\partial\xi}U(1,\xi,\tau) + \frac{\partial}{\partial\rho}W(1,\xi,\tau) = 0 \tag{8}$$

2. Solving a vector one-dimensional boundary problem. In order to reduce the problem to the one-dimensional one, the finite *sin*− and *cos*− Fourier integral transforms with regard of the variable ξ and Laplace integral transformation with regard of the variable τ are applied successively to the differential equations (5) and boundary conditions (6)-(8)

$$\begin{bmatrix}U_\lambda(\rho,\tau)\\ W_\lambda(\rho,\tau)\end{bmatrix} = \int\limits_0^1 \begin{bmatrix}U(\rho,\xi,\tau)\cos\lambda_n\xi\\ W(\rho,\xi,\tau)\sin\lambda_n\xi\end{bmatrix}d\xi,\quad \begin{matrix}n=0,1,2,...\\ n=1,2,...\end{matrix}\quad \lambda_n = \pi n = \lambda$$

$$\begin{bmatrix}U_{\lambda p}(\tau)\\ W_{\lambda p}(\tau)\end{bmatrix} = \int\limits_0^\infty \begin{bmatrix}U_\lambda(\rho,\tau)\\ W_\lambda(\rho,\tau)\end{bmatrix}e^{-pt}d\tau$$

As a result, equations (5) can be written

$$\rho^{-1}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}U_{\lambda p}(\rho)\right]+\frac{2}{\kappa+1}\lambda_*\frac{\partial}{\partial\rho}W_{\lambda p}(\rho)-\rho^{-2}U_{\lambda p}(\rho)-\frac{\kappa-1}{\kappa+1}\lambda_*^2U_{\lambda p}(\rho)- \\ -\frac{\kappa-1}{\kappa+1}p^2U_{\lambda p}(\rho)=0,\ 1<\rho<\infty$$

$$\rho^{-1}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}W_{\lambda p}(\rho)\right]-\rho^{-1}\frac{2}{\kappa-1}\lambda_*\frac{\partial}{\partial\rho}\left[\rho U_{\lambda p}(\rho)\right]-\frac{\kappa+1}{\kappa-1}\lambda_*^2W_{\lambda p}(\rho)- \\ -p^2W_{\lambda p}(\rho)=0,\ \lambda_*=\lambda\alpha \tag{9}$$

During this operation the boundary conditions (6) are automatically satisfied, and conditions (7), (8) have the form

$$U'_{\lambda p}(1)+\frac{3-\kappa}{1+\kappa}\left[U_{\lambda p}(1)+\lambda_*W_{\lambda p}(1)\right]=aG^{-1}\frac{\kappa-1}{\kappa+1}P_{\lambda p}$$

$$W'_{\lambda p}(1)-\lambda_*U_{\lambda p}(1)=0,\ P_{\lambda p}=\int_0^\infty\left(\int_0^1P(\xi,\tau)\cos\lambda_n\xi d\xi\right)e^{-p\tau}d\tau \tag{10}$$

For solving a one-dimensional boundary value problem (9), (10) a second-order matrix differential operator and the unknown vector of displacements' transformations are set

$$L_2=\begin{pmatrix}\rho^{-1}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}\right]-\rho^{-2}-\frac{\kappa-1}{\kappa+1}\left(\lambda_*^2+p^2\right) & \frac{2}{\kappa+1}\lambda_*\frac{\partial}{\partial\rho} \\ -\frac{2}{\kappa-1}\lambda_*\rho^{-1}\frac{\partial}{\partial\rho}\left[\rho\right] & \rho^{-1}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}\right]-\frac{\kappa+1}{\kappa-1}\lambda_*^2-p^2\end{pmatrix}$$

$$\mathbf{y}(\rho)=\begin{pmatrix}U_{\lambda p}(\rho)\\W_{\lambda p}(\rho)\end{pmatrix}$$

Let's set up the boundary functional corresponding to the boundary conditions (10)

$$\mathrm{U}\left[\mathbf{y}(1)\right]=\mathbf{A}\cdot\mathbf{y}(1)+\mathbf{I}\cdot\mathbf{y}'(1),\ \mathbf{A}=\begin{pmatrix}\frac{3-\kappa}{1+\kappa} & \frac{3-\kappa}{1+\kappa}\lambda_*\\-\lambda_* & 0\end{pmatrix},\ \mathbf{I}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

In these notations the boundary value problem (9), (10) is written down in a next form [12]

$$\mathbf{L}_2\mathbf{y}(\rho)=\mathbf{f}(\rho),\ 1<\rho<\infty,\ \mathbf{U}\left[\mathbf{y}(1)\right]=\gamma \tag{11}$$

$$\mathbf{f}(\boldsymbol{\rho})=\begin{pmatrix}0\\0\end{pmatrix},\ \boldsymbol{\gamma}=\begin{pmatrix}\boldsymbol{aG^{-1}\frac{\kappa-1}{\kappa+1}P_{\lambda p}}\\\mathbf{0}\end{pmatrix}$$

In order to get a general decreasing solution when $\rho\to\infty$ of the vector homogeneous equation in (11), the solution of the matrix differential equation

$$L_2\mathbf{Y}(\rho)=0,\ 1<\rho<\infty \tag{12}$$

should be constructed previously.

With the help of the auxiliary matrix

$$\mathbf{H}(\rho,\xi)=\begin{pmatrix} H_1^{(1)}(\rho\xi) & 0\\ 0 & H_0^{1}(\rho\xi)\end{pmatrix}$$

where $H_m^{(1)}(z)$ is the Hankel first order function, $m=0,1$, an important relationship has been proven [10]

$$L_2\mathbf{H}(\rho,\xi)=-\mathbf{H}(\rho,\xi)\cdot\mathbf{M}(\xi),$$
$$\mathbf{M}(\xi)=\begin{pmatrix}\xi^2+\frac{\kappa-1}{\kappa+1}\left(\lambda_*^2+p^2\right) & \frac{2}{\kappa+1}\xi\lambda_*\\ \frac{2}{\kappa-1}\xi\lambda_* & \xi^2+\frac{\kappa+1}{\kappa-1}\lambda_*^2+p^2\end{pmatrix} \tag{13}$$

The inverse matrix $\mathbf{M}(\xi)$ for has the form

$$\mathbf{M}^{-1}(\xi)=\frac{1}{\det\mathbf{M}}\begin{pmatrix}\xi^2+\frac{\kappa+1}{\kappa-1}\lambda_*^2+p^2 & -\frac{2}{\kappa+1}\xi\lambda_*\\ -\frac{2}{\kappa-1}\xi\lambda_* & \xi^2+\frac{\kappa-1}{\kappa+1}\left(\lambda_*^2+p^2\right)\end{pmatrix}$$

$$\det\mathbf{M}=\left[\xi-i\sqrt{\lambda_*^2+p^2}\right]\left[\xi+i\sqrt{\lambda_*^2+p^2}\right]\left[\xi-i\sqrt{\lambda_*^2+\tfrac{\kappa-1}{\kappa+1}p^2}\right]\times$$
$$\times\left[\xi+i\sqrt{\lambda_*^2+\tfrac{\kappa-1}{\kappa+1}p^2}\right]$$

Further, with the help of the equality (13), one can be convinced that the solution of the matrix equation (12) is

$$\mathbf{Y}(\rho)=\frac{1}{2\pi i}\int\limits_C\mathbf{H}(\rho,\xi)\cdot\mathbf{M}^{-1}(\xi)d\xi,$$

where C is the closed loop covering the origin and two poles of the first multiplicity $\xi=i\sqrt{\lambda_*^2+p^2}$, $\xi=i\sqrt{\lambda_*^2+\frac{\kappa-1}{\kappa+1}p^2}$ lying in the upper half-plane. Applying the methods of contour integration, the matrix is derived

$$\mathbf{Y}(\rho)=\tfrac{1}{2p^2}\begin{pmatrix} i\cdot\frac{\kappa+1}{\kappa-1}\frac{\lambda_*^2}{\delta_1}\cdot H_1^{(1)}(i\rho\delta_1) & \lambda_*\cdot H_1^{(1)}(i\rho\delta_1)\\ \frac{\kappa+1}{\kappa-1}\lambda_*\cdot H_0^{(1)}(i\rho\delta_1) & -i\delta_1\cdot H_0^{(1)}(i\rho\delta_1)\end{pmatrix}+$$
$$+\tfrac{1}{2p^2}\begin{pmatrix} -i\cdot\frac{\kappa+1}{\kappa-1}\delta_2\cdot H_1^{(1)}(i\rho\delta_2) & -\lambda_*\cdot H_1^{(1)}(i\rho\delta_2)\\ -\frac{\kappa+1}{\kappa-1}\lambda_*\cdot H_0^{(1)}(i\rho\delta_2) & i\frac{\lambda_*^2}{\delta_2}\cdot H_0^{(1)}(i\rho\delta_2)\end{pmatrix}$$

where

$$\delta_1=\sqrt{\lambda_*^2+p^2}\quad \delta_2=\sqrt{\lambda_*^2+\frac{\kappa-1}{\kappa+1}p^2} \tag{14}$$

which was constracted using the residue theorem.

Taking into account the results in [12] and the range of the parameter $1 < \rho < \infty$, a decreasing solution of the matrix equation is constructed

$$\mathbf{Y}_{\lambda p}(\rho) = \tfrac{1}{p^2}\begin{pmatrix} -i\cdot\frac{\kappa+1}{\kappa-1}\frac{\lambda_*^2}{\delta_1}\cdot K_1(\rho\delta_1) & -\lambda_*\cdot K_1(\rho\delta_1) \\ -i\cdot\frac{\kappa+1}{\kappa-1}\lambda_*\cdot K_0(\rho\delta_1) & -\sqrt{\lambda_*^2+p^2}\cdot K_0(\rho\delta_1)\end{pmatrix} + \\ +\tfrac{1}{p^2}\begin{pmatrix} i\cdot\frac{\kappa+1}{\kappa-1}\delta_2\cdot K_1(\rho\delta_2) & \lambda_*\cdot K_1(\rho\delta_2) \\ i\cdot\frac{\kappa+1}{\kappa-1}\lambda_*\cdot K_0(\rho\delta_2) & \frac{\lambda_*^2}{\delta_2}\cdot K_0(\rho\delta_2)\end{pmatrix}$$

where $K_m(z)$ is the Macdonald function, $m = 0, 1$.

The solution of the one-dimensional problem (11) is written in the form [12]

$$\mathbf{y}(\rho) = \mathbf{Y}_{\lambda p}(\rho)\cdot\begin{pmatrix} iC_0 \\ C_1\end{pmatrix}$$

The reality of the solution's values (15) is guaranteed by the special choice of constants C_0, C_1, which can be found from the boundary conditions (10). It leads to the linear system of equations

$$\begin{cases} a_{11}C_0 + a_{12}C_1 = 0 \\ a_{21}C_0 + a_{22}C_1 = aG^{-1}\frac{\kappa-1}{\kappa+1}p^2\cdot P_{\lambda p}\end{cases}$$

$$a_{11} = \frac{\kappa+1}{\kappa-1}\left\{-\frac{\lambda_*(2\lambda_*^2+p^2)}{\delta_1}K_1(\delta_1) + 2\lambda_*\delta_2 K_1(\delta_2)\right\}$$

$$a_{12} = (2\lambda_*^2+p^2)K_1(\delta_1) - 2\lambda_*^2 K_1(\delta_2)$$

$$a_{21} = \frac{\kappa+1}{\kappa-1}\left\{-\frac{1}{2}\lambda_*^2K_2(\delta_1) + \frac{1}{2}\delta_2^2K_2(\delta_2) + \frac{3-\kappa}{\kappa+1}\left[\frac{\lambda_*^2}{\delta_1}K_1(\delta_1) - \delta_2K_1(\delta_2)\right] + \right. \\ \left. +\frac{5-3\kappa}{2(\kappa+1)}\lambda_*^2K_0(\delta_1) - \left(\frac{5-3\kappa}{2(\kappa+1)}\lambda_*^2 - \frac{\kappa-1}{2(\kappa+1)}p^2\right)K_0(\delta_2)\right\}$$

$$a_{22} = -\tfrac{1}{2}\lambda_*\delta_1K_2(\delta_1) - \tfrac{1}{2}\lambda_*\delta_2K_2(\delta_2) + \tfrac{3-\kappa}{\kappa+1}\lambda_*\left[-K_1(\delta_1) + K_1(\delta_2)\right] - \\ -\tfrac{5-3\kappa}{2(\kappa+1)}\lambda_*\delta_1K_0(\delta_1) - \left(\tfrac{5-3\kappa}{2(\kappa+1)}\lambda_*^2 - \tfrac{\kappa-1}{2(\kappa+1)}p^2\right)\tfrac{\lambda_*}{\delta_2}K_0(\delta_2)$$

where the known derivatives' formulas of special functions [23] were used

$$\frac{\partial}{\partial\rho}K_0(a\rho) = -aK_1(a\rho), \quad \frac{\partial}{\partial\rho}K_1(a\rho) = -\frac{1}{2}a\left[K_0(a\rho) + K_2(a\rho)\right]$$

The coefficients were found in the form

$$C_0 = -\tfrac{a}{\det}G^{-1}\tfrac{\kappa-1}{\kappa+1}p^2\cdot P_{\lambda p}\cdot a_{12},\ C_1 = \tfrac{a}{\det}G^{-1}\tfrac{\kappa-1}{\kappa+1}p^2\cdot P_{\lambda p}\cdot a_{11}, \\ det = a_{11}a_{22} - a_{12}a_{21}$$

The solution of the one-dimensional value problem (11) in transformation domain was constructed with their help

$$U_{\lambda p}(\rho) = \frac{a}{G} P_{\lambda p} \frac{\delta_2}{\Delta} \left[-2 \left(\lambda_*^2 + p^2 \right) K_1(\rho\delta_1) K_1(\delta_2) + (2\lambda_*^2 + p^2) K_1(\rho\delta_2) K_1(\delta_1) \right]$$

$$W_{\lambda p}(\rho) = \frac{a}{G} P_{\lambda p} \frac{\lambda_*}{\Delta} \left[2\delta_1\delta_2 K_0(\rho\delta_1) K_1(\delta_2) - (2\lambda_*^2 + p^2) K_0(\rho\delta_2) K_1(\delta_1) \right] \quad (16)$$

where

$$\begin{gathered} \Delta = \tfrac{\kappa+1}{\kappa-1} \left(\lambda_*^2 + \tfrac{1}{2}p^2 \right) \delta_2^2 K_1(\delta_1) K_2(\delta_2) - \tfrac{\kappa+1}{\kappa-1} \lambda_*^2 \delta_1 \delta_2 K_1(\delta_2) K_2(\delta_1) - \\ \tfrac{3-\kappa}{\kappa-1} p^2 \delta_2 K_1(\delta_1) K_1(\delta_2) + \left(\lambda_*^2 + \tfrac{1}{2}p^2 \right) \left(\tfrac{5-3\kappa}{\kappa-1} \lambda_*^2 - p^2 \right) K_1(\delta_1) K_0(\delta_2) + \\ + \tfrac{5-3\kappa}{\kappa-1} \lambda_*^2 \delta_1 \delta_2 K_1(\delta_2) K_0(\delta_1) \end{gathered} \quad (17)$$

3. The final formulas construction. In order to get the solution of initial problem (1-3), the inverse integral transformations should be applied

$$U_p(\rho,\xi) = U_{0,p}(\rho) + 2\sum_{n=1}^{\infty} U_{\lambda p}(\rho) \cos\lambda_n\xi, \quad W_p(\rho,\xi) = 2\sum_{n=1}^{\infty} W_{\lambda p}(\rho) \sin\lambda_n\xi.$$

The function $U_{0p}(\rho)$ can be found from the following one-dimensional value problem (as $\lambda_0 = 0, W_{\lambda 0p}(\rho) = W_0(\rho) = 0$)

$$\rho\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}U_{0p}(\rho)\right] - U_{0p}(\rho) = 0, \quad \frac{\partial}{\partial\rho}U_{0p}(1) + \frac{3-\kappa}{\kappa+1}U_{0p}(1) = aG^{-1}\frac{\kappa-1}{\kappa+1}P_{0p}.$$

It has the form

$$U_{0p}(\rho) = \frac{a}{4G}\rho \int_0^1 p(h\xi)d\xi. \quad (18)$$

The field of the initial displacements of the infinite elastic layer with the cylindrical cavity is derived

$$U(\rho,\xi,\tau) = \frac{a}{G}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left[U_{0p}(\rho) + 2\sum_{n=1}^{\infty} F_1(\rho) P_{\lambda p} \frac{\delta_2}{\Delta}\cos\lambda_n\xi\right] e^{p\tau}dp$$

$$W(\rho,\xi,\tau) = \frac{a}{G}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left[2\sum_{n=1}^{\infty} F_2(\rho) P_{\lambda p} \frac{\lambda_*}{\Delta}\sin\lambda_n\xi\right] e^{p\tau}dp \quad (19)$$

The normal stress can be constructed with their help by the formula [21]

$$\sigma_\xi = Ga^{-1}\frac{3-\kappa}{\kappa-1}\left[\frac{\partial}{\partial\rho}U(\rho,\xi,\tau) + \rho^{-1}U(\rho,\xi,\tau) + \frac{1+\kappa}{3-\kappa}\alpha\frac{\partial}{\partial\xi}W(\rho,\xi,\tau)\right]$$

4. The subcase of steady-state oscillations. The case of steady-state oscillations is considered below. With this aim the substitution $p = i\omega$, $p^2 = -\omega^2$ was made (p - Laplace transform parameter, ω - circular frequency of steady-state oscillations). Taking into account formula (18) and putting in consideration load of constant intensity $P(\xi) = 1$ in formula (10), one can get the following expression instead of (19)

$$U(\rho, \xi; \omega) = \frac{a}{4G}\rho + \frac{2a}{G}\sum_{n=1}^{\infty}\frac{\cos\lambda_n\xi \cdot \sin\lambda_n}{\lambda_n}\frac{\Delta_2}{\Delta_3}F_n^{(1)}(\rho;\omega) \tag{20}$$

$$W(\rho, \xi; \omega) = \frac{2a}{G}\sum_{n=1}^{\infty}\frac{\sin\lambda_n\xi \cdot \sin\lambda_n}{\lambda_n}\frac{\lambda_*}{\Delta_3}F_n^{(2)}(\rho;\omega)$$

where

$$\Delta_1 = \sqrt{\lambda_*^2 - \omega^2} \quad \Delta_2 = \sqrt{\lambda_*^2 - \frac{\kappa - 1}{\kappa + 1}\omega^2}$$

$$\Delta_3 = \tfrac{\kappa+1}{\kappa-1}\left(\lambda_*^2 - \tfrac{1}{2}\omega^2\right)\Delta_2^2 K_1(\Delta_1)K_2(\Delta_2) - \tfrac{\kappa+1}{\kappa-1}\lambda_*^2\Delta_1\Delta_2 K_1(\Delta_2)K_2(\Delta_1)+$$
$$+\tfrac{3-\kappa}{\kappa-1}\omega^2\Delta_2 K_1(\Delta_1)K_1(\Delta_2) + \left(\lambda_*^2 - \tfrac{1}{2}\omega^2\right)\left(\tfrac{5-3\kappa}{\kappa-1}\lambda_*^2 + \omega^2\right)K_1(\Delta_1)K_0(\Delta_2)+$$
$$+\tfrac{5-3\kappa}{\kappa-1}\lambda_*^2\Delta_1\Delta_2 K_1(\Delta_2)K_0(\Delta_1)$$

$$F_n^{(1)}(\rho;\omega) = -2\left(\lambda_*^2 - \omega^2\right)K_1(\rho\Delta_1)K_1(\Delta_2) + \left(2\lambda_*^2 - \omega^2\right)K_1(\rho\Delta_2)K_1(\Delta_1)$$

$$F_n^{(2)}(\rho;\omega) = 2\Delta_1\Delta_2 K_0(\rho\Delta_1)K_1(\Delta_2) - \left(2\lambda_*^2 - \omega^2\right)K_0(\rho\Delta_2)K_1(\Delta_1)$$

The normal stress of the layer is derived on the base of displacements (20)

$$\sigma_\xi(\rho, \xi, \omega) = \frac{3-\kappa}{2(\kappa - 1)}\left[1 + \sum_{n=1}^{\infty}\frac{\cos\lambda_n\xi \cdot \sin\lambda_n}{\lambda_n}\frac{1}{\Delta_3}F_n(\rho, \omega)\right] \tag{21}$$

$$F_n(\rho, \omega) = -2\left(\lambda_*^2 - \omega^2\right)\Delta_2\rho^{-1}K_1(\rho\Delta_1)K_1(\Delta_2)+$$
$$+\left(2\lambda_*^2 - \omega^2\right)\Delta_2\rho^{-1}K_1(\rho\Delta_2)K_1(\Delta_1) + \left(\lambda_*^2 - \omega^2\right)\Delta_1\Delta_2 K_2(\rho\Delta_1)K_1(\Delta_2)-$$
$$-\left(\lambda_*^2 - \tfrac{1}{2}\omega^2\right)\Delta_2^2 K_0(\rho\Delta_2)K_1(\Delta_1) + \left(\tfrac{5+\kappa}{3-\kappa}\lambda_*^2 - \omega^2\right)\Delta_1\Delta_2 K_0(\rho\Delta_1)K_1(\Delta_2)-$$
$$-\left(\lambda_*^2 - \tfrac{1}{2}\omega^2\right)\left(\tfrac{5+\kappa}{3-\kappa}\lambda_*^2 - \tfrac{\kappa-1}{\kappa+1}\omega^2\right)K_0(\rho\Delta_2)K_1(\Delta_1)$$

$$\lambda_* = \lambda_n \cdot \alpha = \pi n \cdot \alpha,\ \alpha = \frac{a}{h},\ \kappa = 3 - 4\mu$$

5. Discussion and numerical results. The normal stress on the lower face of the layer $\xi = 0,\ 1 < \rho < \infty$, was investigated, depending on different mechanical characteristics: Poisson's ratio $\mu = 1/3$ or $\mu = 1/4$, ratio of cavity radius to layer thickness $\alpha = a/h$, different variants of natural oscillation frequencies $\omega = 0.1, 0.3, 0.5, 1, 3$. The possibility of the appearance of tensile stress on the lower face of the layer was considered. The dynamic load of constant intensity was set on the cylindrical surface of the cavity.

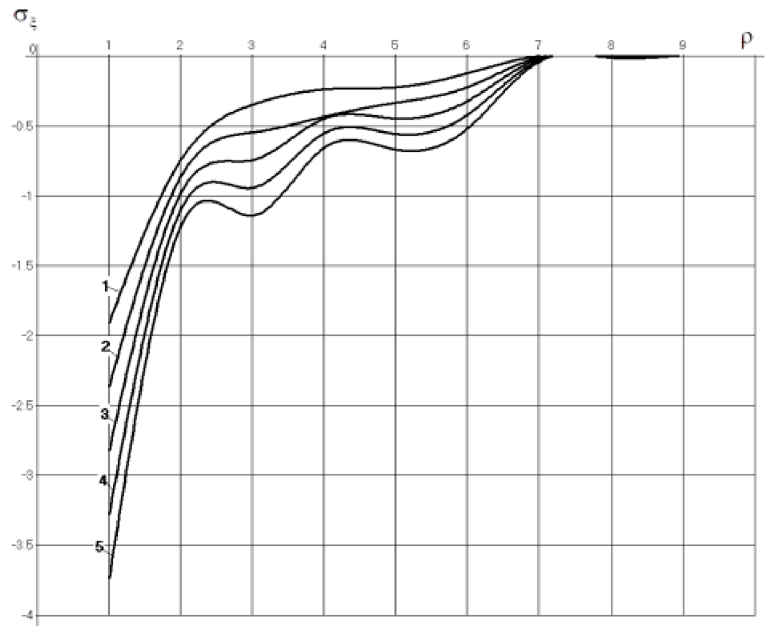

Fig. 2. The normal stress on the lower layer's face

3. Conclusion

The dynamical problem's solution of the elasticity for the infinite layer with a cylindrical cavity was derived, when on the faces of the layer the ideal contact conditions are given and the cavity's surface is under the influence of the normal dynamic tensile force, applied at the initial moment of time. Applying the integral transform method directly to the movement equations reduces the initial problem to the one-dimensional vector problem. The last one was solved exactly using the matrix differential calculus.

It should be noted that similar vector boundary problem can be obtained for the elastic layer weakened by a cylindrical inclusion $0 \leq \rho \leq a$, when different kinds of the boundary conditions at a defect's surface and the layer's faces are set.

At the subcase of the ideal contact conditions on a defect's surface or on the edges, the proposed approach makes it possible to obtain an exact solution of the problem.

When some of the layer's face is rigidly fixed, it leads the initial problem to an integral singular equation with respect to an unknown displacement derivative, so the approximate solution will be constructed. If a vector differential equation is inhomogeneous one, the matrix Green's function and the fundamental matrix should be found.

It is worth noting that the difficulties connecting with the integral Laplace transform inversing exist, so it is often possible to investigate just a case of steady-state oscillations.

Фесенко Г. О.

Точний розв'язок динамічної задачі для нескінченного шару з циліндричним отвором

Резюме

Побудовано хвильове поле нескінченного пружного шару, послабленого циліндричним отвором. Умови ідеального контакту задано на верхній та нижній гранях шару. Нормальне динамічне розтягувальне навантаження діє на поверхні циліндричного отвору в початковий момент часу. Інтегральні перетворення Лапласа та скінченні $sin-$ та $cos-$ Фур'є застосовано послідовно до осесиметричних рівнянь руху та до граничних умов, на відміну традиційним підходам, коли інтегральні перетворення застосовуються до подання розв'язків через гармонічні та бігармонічні функції. Це приводить до одновимірної векторної однорідної крайової задачі відносно невідомих трансформант переміщень. Задачу розв'язано за допомогою матричного диференціального числення. Поле вихідних переміщень знайдено після застосування обернених інтегральних перетворень. Побудовано нормальне напруження на гранях пружного шару.
Ключові слова: точний розв'язок, динамічне навантаження, циліндричний отвір, інтегральні перетворення.

Фесенко А. А.

Точное решение динамической задачи для бесконечного слоя с цилиндрическим отверстием

Резюме

Построено волновое поле бесконечного упругого слоя, ослабленного цилиндрическим отверстием. Условия идеального контакта заданы на гранях слоя. Нормальная динамическая растягивающая нагрузка действует на поверхности цилиндрического отверстия в начальный момент времени. Интегральные преобразования Лапласа и конечные $sin-$ и $cos-$ Фурье применены последовательно к осесимметричным уравнениям движения и к граничным условиям, в отличие от традиционных подходов, когда интегральные преобразования применяются к представлениям решений через гармоничные и бигармонические функции. Это приводит к одномерной векторной однородной краевой задаче относительно неизвестных трансформант перемещений. Задача решена с помощью матричного дифференциального исчисления. Поле исходных перемещений найдено после применения обратных интегральных преобразований. Построено нормальное напряжение на гранях упругого слоя.
Ключевые слова: точное решение, динамическая нагрузка, цилиндрическая полость, интегральные преобразования.

References

1. Popov, G. Ya. (2013). *An exact solution of the elasticity theory problem for an infinite layer weakened by a cylindric cavity. Dokladu RUN,* Vol. 451 (5), P. 1-4.

2. Menshykov, O., Menshykova, M. & Vaysfeld, N. (2017). *Exact analytical solution for a pie shaped wedge thick plate under oscillating load. Acta Mechanica,* Vol. 228 (12), P. 4435-4450.

3. Malitz, P. Ya., Privarnikov, A. K. (1971). *The application of Weber-type transformations to the solution of elasticity problems for layered media with a cylindrical hole. J. Voprosu prochnosty i plastichnossty,* P. 56 - 64.

4. Arutunyan, N. H., Abramyan, B. L. (1969). *Some axisymmetric problems for a half-space and an elastic layer with a vertical cylindrical notch. J. Izv. AN Arm. SSR. Mekhanica,* Vol. 22 (3), P. 3-13.

5. Yahnioglu, N., Babuscu Yesil, U. (2009). *Forced vibration of an initial stressed rectangular composite thick plate with a cylindrical hole. ASME International Mechanical Engineering Congress and Exposition* IMECE09, Lake BuenaVista, Florida, USA.

6. Jain, N. K., Mittal N. D. (2008). *Finite element analysis for stress concentration and deflection in isotopic, orthotropic and laminated composite plates with central circular hole under transverse static load. Materials Science and Engineering,* Vol. 498. P. 115-124.

7. Guz', A. N. (1962). *Approximate method for calculation of the stress concentrations around curvilinear holes in shells. Prikl. Mekh,* Vol. 2 (6), P. 605-612.

8. Bobyleva, T. (2016). *Approximate method of calculating stresses in layered array. Procedia Engineering,* Vol. 153, P. 103-106.

9. Vorovich, I. I., Babeshko, V. A. (1979). *Dynamic mixed problems in elasticity theory for nonclassical regions,* Nauka: Moscow, 320 p.

10. Bardzokas, D. I., Kushnir, D. V., Filshtinskii, L. A. (2009). *Dynamic problems of the theory of elasticity for layers and semilayers with cavities. J. Acta Mech.,* Vol. 208, P. 81-95.

11. Grinchenko, V. T., Meleshko, V. V. (1981). *Harmonic vibrations and waves in elastic bodies.* Kiev: Naukova Dumka, 284 p.

12. Kubenko, V. D. (1965). *Propagation of elastic waves from a circular hole in an anisotropic inhomogeneous plate. Prikl. Mekh.,* Vol. 1 (2), P. 25-33.

13. Panasyuk, N. N. (1978). *Action of a plane step elastic wave on a spherical cavity. Waves in Continuous Media.* Kiev: Naukova Dumka, P. 79-85.

14. Baoping Hei, Zailin Yang, Yao Wang. (2016). *Dynamic analysis of elastic waves by an arbitrary cavity in an inhomogeneous medium with density variation. Mathematics and Mechanics of Solids.* Vol 21 (8), P. 931-940.

15. Linton, C. M., Thompson, I. (2018). *Elastic waves trapped above a cylindrical cavity. SIAM J. Appl. Math.,* Vol. 78 (4), P. 2083-2104.

16. Zhou, Y., Zheng, R.-Y., Liu, G.-B. (2011). *Dynamic response of elastic layer on transversely isotropic saturated soil to train load. Yantu Lixue/Rock and Soil Mechanics,* Vol. 32 (2), P. 604-610.

17. Zhuk, A. P., Kubenko, V. D., Zhuk, Ya. A. (2012). *Acoustic radiation force on a spherical particle in a fluid-filled cavity. J. Acoust. Soc. America,* Vol. 132 (4), P. 2189-2197.

18. Ming Tao, Ao Ma, Wenzhuo Cao, Xibing Li. (2017). *Dynamic response of pre-stressed rock with a circular cavity subject to transient loading. International Journal of Rock Mechanics and Mining Sciences,* Vol. 99, P. 1-8.

19. Gaoa, M. (2016). *An exact solution for three-dimensional (3D) dynamic response of a cylindrical lined tunnel in saturated soil to an internal blast load. Soil Dynamics and Earthquake Engineering,* Vol. 90, P. 32-37.

20. Yih-Hsing Pao. (1983). *Elastic waves in solids. J. Appl. Mech,* Vol. 50 (4), P. 1152-1164.

21. Novazkiy, W. (1975). *The theory of elasticity.* Moscow: Mir, 872 p.

22. Popov, G. Ya., Abdimanapov, S. A., Efimov, V. V. (1999). *Green's functions and matrix of one-dimensional boundary value problems.* Almati: Rauan, 113 p.

23. Gradshtein, I., Rygik, L. (1963). *The tables of integrals, series and products.* Moscow: Nauka, 1100 p.