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COMPARISON THEOREM FOR NEUTRAL STOCHASTIC INTEGRO-DIFFERENTIAL EQUATIONS IN HILBERT SPACE

In the present paper, we will discuss a comparison result for solutions to the Cauchy problems for two stochastic differential equations with delay. On this subject number of authors have obtained their comparison results. We deal with the Cauchy problems for two neutral stochastic integro-differential equations. Except transient- (or drift-) and diffusion-coefficients, our equations include also one integro-differential term. Basic difference of our case from the case of all earlier investigated problems is presence of this term. We intoduce a concept of solutions to our problems and prove the comparison theorem for them. According to our result, under certain assumptions on coefficients of equations under consideration, their solutions depend on the transient-coefficients in a monotone way.
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Introduction. In the given paper the fallowing Cauchy problems for two neutral stochastic integro-differential equations

$$d\left(u_i(t,x) + \int\limits_{\mathbb{R}^d} b_i(t,x,u_i(\alpha(t),\xi),\xi)d\xi\right) = f_i(t,u_i(\alpha(t),x),x)dt \\ + \sigma(t,x)d\beta(t),\ 0 < t \leqslant T,\ x \in \mathbb{R}^d,\ i \in \{1,2\}, \tag{1}$$

$$u_i(t,x) = \phi_i(t,x),\ -r \leqslant t \leqslant 0,\ x \in \mathbb{R}^d,\ r > 0,\ i \in \{1,2\}, \tag{1*}$$

are studied, where $T > 0$ is fixed, β is one-dimensional Brownian motion, $f_i \colon [0,T] \times \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $i \in \{1,2\}$, $\sigma \colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$ and $b_i \colon [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $i \in \{1,2\}$, are some given functions to be specified later, $\phi_i \colon [-r,0] \times \mathbb{R}^d \to \mathbb{R}$, $i \in \{1,2\}$, are initial-datum functions, $\alpha \colon [0,T] \to [-r,+\infty)$ is a delay function. For solutions u_1 and u_2 of these problems a comparison theorem is proved. According to the obtained result, if $f_1 \geqslant f_2$, then $u_1 \geqslant u_2$ with probability one.

A comparison problem for solutions to stochastic differential equations in finite-dimensional case has firstly arised in [9]. A comparison theorem for equation of the form $d\xi(t) = f(t,\xi(t))dt + \sigma(t,\xi(t))d\beta(t)$ has been proved in this work by A. V. Skorokhod. According to this theorem, under certain assumptions, a solution of the equation above is monotonously non-decreasing function, depending on drift-coefficient f. A more general presentation of the comparison theorem is given in [7], [8]. Variations of these results have been proposed in [1] – [6]. The aim of the given work was to prove the comparison theorem for solutions of problem (1) – (1*).

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Main Results

1. Formulation of the problem Throughout the paper $(\Omega, \mathcal{F}, \mathbf{P})$ will note a complete probability space. Let
$\{\mathcal{F}_t, t \geqslant 0\}$ be a normal filtration on $\mathcal{F}$. From now on $L_2(\mathbb{R}^d)$ will note real Hilbert space with the norm $\|g\|_{L_2(\mathbb{R}^d)} = \left(\int\limits_{\mathbb{R}^d} g^2(x)dx\right)^{\frac{1}{2}}$.

We impose the following conditions

1. $\alpha\colon [0,T] \to [-r, +\infty)$ belongs to $C^1([0,T])$ with $\alpha' \geqslant 1$, $\alpha(t) \leqslant t$.
2. $f_i\colon [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $i \in \{1,2\}$, $\sigma\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$, $b_i\colon [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \to \mathbb{R}$, $i \in \{1,2\}$, are measurable with respect to all of their variables functions.
3. The initial-datum functions $\phi_i(t,x,\omega)\colon [-r,0] \times \mathbb{R}^d \times \Omega \to L_2(\mathbb{R}^d)$, $i \in \{1,2\}$, are $\mathcal{F}_0$-measurable random variables and such that
$$\sup_{-r \leqslant t \leqslant 0} \mathbf{E}\|\phi_i(t,\cdot)\|^2_{L_2(\mathbb{R}^d)} < \infty,\ i \in \{1,2\}.$$
4. b_i, $i \in \{1,2\}$, satisfy the Lipshitz condition in the third argument of the form
$$\begin{gathered} |b_i(t,x,u,\xi) - b_i(t,x,v,\xi)| \leqslant l(t,x,\xi)|u-v|, \\ 0 \leqslant t \leqslant T,\ \{x,\xi\} \subset \mathbb{R}^d,\ \{u,v\} \subset \mathbb{R},\ i \in \{1,2\}, \end{gathered} \tag{2}$$
where $l\colon [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ is such that
$$\sup_{0 \leqslant t \leqslant T} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx < \frac{1}{4}. \tag{3}$$
5. There exists a function $\chi\colon \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$, satisfying the following condition
$$\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d} \chi(x,\xi)d\xi\right)^2 dx < \infty,$$
such that
$$\sup_{0 \leqslant t \leqslant T} |b_i(t,x,0,\xi)| \leqslant \chi(x,\xi),\ 0 \leqslant t \leqslant T,\ x \in \mathbb{R}^d,\ \xi \in \mathbb{R}^d,\ i \in \{1,2\}. \tag{4}$$
6. There exists a function $\eta\colon [0,T] \times \mathbb{R}^d \to [0,\infty)$ with
$$\sup_{0 \leqslant t \leqslant T} \int\limits_{\mathbb{R}^d} \eta^2(t,x)dx < \infty,$$
such that the following linear-growth and Lipschitz conditions are valid for f_i, $i \in \{1,2\}$,
$$|f_i(t,u,x)| \leqslant \eta(t,x) + L|u|,\ 0 \leqslant t \leqslant T,\ u \in \mathbb{R},\ x \in \mathbb{R}^d,\ i \in \{1,2\}, \tag{5}$$
$$|f_i(t,u,x) - f_i(t,v,x)| \leqslant L|u-v|,\ 0 \leqslant t \leqslant T,\ \{u,v\} \subset \mathbb{R},\ x \in \mathbb{R}^d,\ i \in \{1,2\}.$$

7. The following condition is true for σ

$$\sup_{0\leqslant t\leqslant T}\|\sigma(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} < \infty.$$

Let $u \equiv u_i$, $\phi \equiv \phi_i$, $b \equiv b_i$, $f \equiv f_i$, $i \in \{1,2\}$.

Definition 1. *A continuous random process* $u(t,x,\omega)\colon [-r,T]\times\mathbb{R}^d\times\Omega \to L_2(\mathbb{R}^d)$ *is called a* ***solution*** *to* (1) – (1*) *provided*

1. *It is* $\mathcal{F}_t$*-measurable for almost all* $-r \leqslant t \leqslant T$.
2. *It satisfies the following integral equation*

$$\begin{gathered} u(t,x) = \phi(0,x) + \int_{\mathbb{R}^d} b(0,x,\phi(-r,\xi),\xi)d\xi - \int_{\mathbb{R}^d} b(t,x,u(\alpha(t),\xi),\xi)d\xi \\ + \int_0^t f(s,u(\alpha(s),x),x)ds + \int_0^t \sigma(s,x)d\beta(s),\ 0\leqslant t\leqslant T,\ x\in\mathbb{R}^d, \end{gathered} \tag{6}$$

$$u(t,x) = \phi(t,x),\ -r\leqslant t\leqslant 0,\ x\in\mathbb{R}^d,\ r>0.$$

3. *It satisfies the condition*

$$\mathbf{E}\int_0^T \|u(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}dt < \infty.$$

Remark 1. *It is assumed in the definition above that all the integrals from* (6) *are well defined.*

Theorem 1. *Suppose assumptions 1 – 7 hold. Then* (6) *has a unique solution.*

Theorem 2 (comparison theorem)**.** *Suppose assumptions 1 – 7 are satisfied and*

1. *The initial-datum functions are such that*

$$\phi_1(t,x) \geqslant \phi_2(t,x),\ 0\leqslant t\leqslant T,\ x\in\mathbb{R}^d,\ i\in\{1,2\}.$$

2. *The functions* b_i, $i\in\{1,2\}$, *satisfy the conditions*

$$\begin{aligned} &b_1(0,x,\phi_2(-r,\xi),\xi) = b_2(0,x,\phi_2(-r,\xi),\xi),\ \{x,\xi\}\subset\mathbb{R}^d,\\ &b_1(0,x,\phi_1(-r,\xi),\xi) = b_2(0,x,\phi_1(-r,\xi),\xi),\ \{x,\xi\}\subset\mathbb{R}^d,\\ &b_1(0,x,\phi_1(-r,\xi),\xi) = b_1(0,x,\phi_2(-r,\xi),\xi),\ \{x,\xi\}\subset\mathbb{R}^d,\\ &b_1(t,x,u,\xi) \leqslant b_2(t,x,u,\xi),\ 0\leqslant t\leqslant T,\ \{x,\xi\}\subset\mathbb{R}^d,\ u\in\mathbb{R}. \end{aligned}$$

3. *For the functions* f_i, $i\in\{1,2\}$, *the following conditions are fulfilled*

$$f_1(t,u,x) \geqslant f_2(t,u,x),\ 0\leqslant t\leqslant T,\ u\in\mathbb{R},\ x\in\mathbb{R}^d.$$

Let one of the following conditions be true

M1. b_1 *is monotonously non-increasing,* f_1 *is monotonously non-decreasing with respect to* u*.*

M2. b_2 *is monotonously non-increasing,* f_2 *is monotonously non-decreasing with respect to* u*.*

Then for all $0 \leqslant t \leqslant T$ *the solutions of* (1) – (1*) *satisfy the inequality*

$$u_1(t,x) \geqslant u_2(t,x),\ x \in \mathbb{R}^d,$$

with probability one.

1. Proof of the theorem 1. In order to prove existence and uniqueness of solution to (6) we use the method of successive approximations. The idea of the proof is to construct a sequence of approximations, which converges to the solution u. From now on x is supposed to be fixed. Let

$$u^{(0)}(t,\cdot) = \phi(0,\cdot),\ 0 < t \leqslant T, \tag{7}$$

$$u^{(0)}(t,\cdot) = \phi(t,\cdot),\ -r \leqslant t \leqslant 0, \tag{7*}$$

and for $n \in \{1, 2, \dots\}$ define $u^{(n)}$ as

$$u^{(n)}(t,\cdot) = \phi(0,\cdot) + \int\limits_{\mathbb{R}^d} b(0,\cdot,\phi(-r,\xi),\xi)d\xi - \int\limits_{\mathbb{R}^d} b(t,\cdot,u^{(n-1)}(\alpha(t),\xi),\xi)d\xi$$
$$+ \int\limits_0^t f(s,u^{(n-1)}(\alpha(s),\cdot),\cdot)ds + \int\limits_0^t \sigma(s,\cdot)d\beta(s),\ 0 < t \leqslant T, \tag{8}$$

$$u^{(n)}(t,\cdot) = \phi(t,\cdot),\ -r \leqslant t \leqslant 0. \tag{8*}$$

1.1. Firstly let us choose a small $0 \leqslant T_1 \leqslant T$ and prove that $\sup\limits_{0\leqslant t\leqslant T_1} \mathbf{E}\|u^{(n)}(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}$ has a bound, independent of n. We obtain

$$\sup_{0\leqslant t\leqslant T_1} \mathbf{E}\|u^{(n)}(t,\cdot)\|^2_{L_2(\mathbb{R}^d)} \leqslant 8\mathbf{E}\|\phi(0,\cdot)\|^2_{L_2(\mathbb{R}^d)} + 8\mathbf{E}\left\|\int\limits_{\mathbb{R}^d} |b(0,\cdot,\phi(-r,\xi),\xi)|d\xi\right\|^2_{L_2(\mathbb{R}^d)}$$
$$+ 2\sup_{0\leqslant t\leqslant T_1} \mathbf{E}\left\|\int\limits_{\mathbb{R}^d} |b(t,\cdot,u^{(n-1)}(\alpha(t),\xi),\xi)|d\xi\right\|^2_{L_2(\mathbb{R}^d)}$$
$$+ 8\sup_{0\leqslant t\leqslant T_1} \mathbf{E}\left\|\int\limits_0^t |f(s,u^{(n-1)}(\alpha(s),\cdot),\cdot)|ds\right\|^2_{L_2(\mathbb{R}^d)} + 8\sup_{0\leqslant t\leqslant T_1} \mathbf{E}\left\|\int\limits_0^t \sigma(s,\cdot)d\beta(s)\right\|^2_{L_2(\mathbb{R}^d)}$$
$$= 8\mathbf{E}\|\phi(0,\cdot)\|^2_{L_2(\mathbb{R}^d)} + \sum_{j=1}^4 S_j,\ 0 < t \leqslant T. \tag{9}$$

From (2) and (4) we have

$$|b(t,\,\cdot\,,u,\xi)| \leqslant |b(t,\,\cdot\,,u,\xi) - b(t,\,\cdot\,,0,\xi)| + |b(t,\,\cdot\,,0,\xi)| \leqslant l(t,\,\cdot\,,\xi)|u| + \chi(\,\cdot\,,\xi),$$
$$0 \leqslant t \leqslant T,\ u \in \mathbb{R},\ \xi \in \mathbb{R}^d.$$

Then we obtain

$$S_1 = 8\mathbf{E}\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d}|b(0,x,\phi(-r,\xi),\xi)|d\xi\right)^2 dx \leqslant 16\mathbf{E}\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d} l(0,x,\xi)\phi(-r,\xi)d\xi\right)^2 dx$$
$$+ 16\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d}\chi(x,\xi)d\xi\right)^2 dx \leqslant 16\left(\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(0,x,\xi)d\xi dx\right)\mathbf{E}\|\phi(-r,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}$$
$$+ 16\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d}\chi(x,\xi)d\xi\right)^2 dx,$$

$$S_2 = 2\sup_{0\leqslant t\leqslant T_1}\mathbf{E}\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d}|b(t,x,u^{(n-1)}(\alpha(t),\xi),\xi)|d\xi\right)^2 dx \leqslant 4\left(\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)$$
$$\times \sup_{0\leqslant t\leqslant T_1}\mathbf{E}\|u^{(n-1)}(\alpha(t),\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} + 4\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d}\chi(x,\xi)d\xi\right)^2 dx. \tag{10}$$

According to properties of α, there exists a point $0 \leqslant t^* \leqslant T_1$, $\alpha(t^*) = 0$. Then

$$\sup_{0\leqslant t\leqslant T_1}\mathbf{E}\|u^{(n-1)}(\alpha(t),\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} \leqslant \sup_{0\leqslant t\leqslant t^*}\mathbf{E}\|u^{(n-1)}(\alpha(t),\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} +$$
$$+ \sup_{t^*\leqslant t\leqslant \alpha(T_1)}\mathbf{E}\|u^{(n-1)}(\alpha(t),\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} \leqslant \sup_{-r\leqslant t\leqslant 0}\mathbf{E}\|\phi(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} +$$
$$+ \sup_{0\leqslant t\leqslant T_1}\mathbf{E}\|u^{(n-1)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}$$

and we get from (10)

$$S_2 \leqslant 4\left(\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right) \times \left(\sup_{-r\leqslant t\leqslant 0}\mathbf{E}\|\phi(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} +\right.$$
$$\left. + \sup_{0\leqslant t\leqslant T_1}\mathbf{E}\|u^{(n-1)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}\right) + 4\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d}\chi(x,\xi)d\xi\right)^2 dx.$$

If t^* does not exist, then $\alpha(t) < 0$ for all t and further conclusions are obvious, because

$$\sup_{0\leqslant t\leqslant T_1}\mathbf{E}\|u^{(n-1)}(\alpha(t),\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} = \sup_{-r\leqslant t\leqslant 0}\mathbf{E}\|\phi(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}.$$

In order to estimate S_3, we take (5) into account and obtain

$$S_3 = 8 \sup_{0\leqslant t\leqslant T_1} \mathbf{E} \int_{\mathbb{R}^d} \left(\int_0^t |f(s,u^{(n-1)}(\alpha(s),x),x)|ds \right)^2 dx \leqslant$$

$$\leqslant 16T_1 \sup_{0\leqslant t\leqslant T_1} \mathbf{E} \int_0^t \int_{\mathbb{R}^d} \left(\eta^2(s,x) + L^2 \big(u^{(n-1)}(\alpha(s),x)\big)^2 \right) dxds \leqslant$$

$$\leqslant 16T_1 \left(T_1 \sup_{0\leqslant t\leqslant T_1} \int_{\mathbb{R}^d} \eta^2(t,x)dx + L^2 \int_{-r}^{\alpha(T_1)} \mathbf{E}\|u^{(n-1)}(s,\cdot)\|^2_{L_2(\mathbb{R}^d)} ds \right) \leqslant$$

$$\leqslant 16T_1^2 \sup_{0\leqslant t\leqslant T_1} \int_{\mathbb{R}^d} \eta^2(t,x)dx + 16L^2T_1 \Bigg(r \sup_{-r\leqslant s\leqslant 0} \mathbf{E}\|\phi(s,\cdot)\|^2_{L_2(\mathbb{R}^d)} +$$

$$+ \int_0^{T_1} \mathbf{E}\|u^{(n-1)}(s,\cdot)\|^2_{L_2(\mathbb{R}^d)} ds \Bigg).$$

For S_4 we conclude

$$S_4 = 8 \sup_{0\leqslant t\leqslant T_1} \int_{\mathbb{R}^d} \int_0^t \left(\sigma^2(s,x)ds \right) dx \leqslant 8 \int_0^{T_1} \|\sigma(s,\cdot)\|^2_{L_2(\mathbb{R}^d)} ds.$$

Let denote

$$S(T_1) = 8\mathbf{E}\|\phi(0,\cdot)\|^2_{L_2(\mathbb{R}^d)} + 16 \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} l^2(0,x,\xi)d\xi dx \right) \mathbf{E}\|\phi(-r,\cdot)\|^2_{L_2(\mathbb{R}^d)} +$$

$$+ 20 \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \chi(x,\xi)d\xi \right)^2 dx + 4 \left(\sup_{0\leqslant t\leqslant T} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx \right) \sup_{-r\leqslant t\leqslant 0} \mathbf{E}\|\phi(t,\cdot)\|^2_{L_2(\mathbb{R}^d)} +$$

$$+ 16T_1^2 \sup_{0\leqslant t\leqslant T_1} \int_{\mathbb{R}^d} \eta^2(t,x)dx + 16rL^2T_1 \sup_{-r\leqslant t\leqslant 0} \mathbf{E}\|\phi(t,\cdot)\|^2_{L_2(\mathbb{R}^d)} +$$

$$+ 8 \int_0^{T_1} \|\sigma(t,\cdot)\|^2_{L_2(\mathbb{R}^d)} dt < \infty.$$

Then from (9) we obtain

$$\sup_{0\leqslant t\leqslant T_1} \mathbf{E}\|u^{(n)}(t,\cdot)\|^2_{L_2(\mathbb{R}^d)} \leqslant S(T_1) + 4 \left(\sup_{0\leqslant t\leqslant T} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx \right) \times \tag{11}$$

$$\times \sup_{0\leqslant t\leqslant T_1} \mathbf{E}\|u^{(n-1)}(t,\cdot)\|^2_{L_2(\mathbb{R}^d)} + 16L^2T_1 \int_0^{T_1} \mathbf{E}\|u^{(n-1)}(t,\cdot)\|^2_{L_2(\mathbb{R}^d)} dt. \tag{12}$$

If $n = 1$, then from (11) we have

$$\sup_{0\leqslant t\leqslant T_1}\mathbf{E}\|u^{(1)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}\leqslant S(T_1)+4\left(\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx\right)\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}$$
$$+16L^2T_1\int\limits_0^{T_1}\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}dt.$$

For an arbitrary $n\in\{2,3,\,\dots\}$ we obtain

$$\sup_{0\leqslant t\leqslant T_1}\mathbf{E}\|u^{(n)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}\leqslant S(T_1)\Bigg[1+4\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx+\dots$$
$$+\left(4\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx\right)^{n-1}\Bigg]+16L^2T_1\int\limits_0^{T_1}S(T_1)\Bigg[1+4\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx+\dots$$
$$+\left(4\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx\right)^{n-2}\Bigg]ds+16L^2T_1\int\limits_0^{T_1}\big(16L^2T_1\big(T_1-s\big)\big)S(T_1)$$
$$\times\Bigg[1+\dots+\left(4\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx\right)^{n-3}\Bigg]ds+\dots$$
$$+16L^2T_1\int\limits_0^{T_1}\frac{\big(16L^2T_1\big(T_1-s\big)\big)^{n-3}}{\big(n-3\big)!}S(T_1)\Bigg[1+4\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx\Bigg]ds$$
$$+\left(4\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx\right)^{n-1}\Bigg[\left(4\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx\right)\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}$$
$$+16L^2T_1\int\limits_0^{T_1}\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}ds\Bigg]+16L^2T_1\left(4\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx\right)^{n-2}$$
$$\times\int\limits_0^{T_1}\Bigg[\left(4\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx\right)\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}+16L^2T_1\int\limits_0^{T_1}\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}ds\Bigg]d\tau$$
$$+\big(16L^2T_1\big)^2\left(4\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx\right)^{n-3}\int\limits_0^{T_1}(T_1-\tau)$$
$$\times\Bigg[\left(4\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx\right)\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}+16L^2T_1\int\limits_0^{T_1}\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}ds\Bigg]d\tau$$

$$
\begin{aligned}
&+\ldots+\big(16L^2T_1\big)^{n-3}\Bigg(4\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\Bigg)^2\int\limits_0^{T_1}\frac{(T_1-\tau)^{n-4}}{(n-4)!}\\
&\times\Bigg[\Bigg(4\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\Bigg)\mathbf{E}\|\phi(0,\cdot)\|^2_{L_2(\mathbb{R}^d)}+16L^2T_1\int\limits_0^{T_1}\mathbf{E}\|\phi(0,\cdot)\|^2_{L_2(\mathbb{R}^d)}ds\Bigg]d\tau\\
&+\big(16L^2T_1\big)^{n-2}\Bigg(4\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\Bigg)\int\limits_0^{T_1}\frac{(T_1-\tau)^{n-3}}{(n-3)!}\\
&\times\Bigg[\Bigg(4\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\Bigg)\mathbf{E}\|\phi(0,\cdot)\|^2_{L_2(\mathbb{R}^d)}+16L^2T_1\int\limits_0^{T_1}\mathbf{E}\|\phi(0,\cdot)\|^2_{L_2(\mathbb{R}^d)}ds\Bigg]d\tau\\
&+\Bigg(4\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\Bigg)^{n-2}16L^2T_1\int\limits_0^{T_1}C(T_1)ds+\Bigg(4\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\Bigg)^{n-3}\\
&\times 16L^2T_1\int\limits_0^{T_1}\big(16L^2T_1(T_1-s)\big)C(T_1)ds+\ldots+\Bigg(4\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\Bigg)^2 16L^2T_1\\
&\times\int\limits_0^{T_1}\frac{\big(16L^2T_1(T_1-s)\big)^{n-4}}{(n-4)!}C(T_1)ds+\Bigg(4\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\Bigg)16L^2T_1\\
&\times\int\limits_0^{T_1}\frac{\big(16L^2T_1(T_1-s)\big)^{n-3}}{(n-3)!}C(T_1)ds+16L^2T_1\int\limits_0^{T_1}\frac{\big(16L^2T_1(T_1-s)\big)^{n-2}}{(n-2)!}C(T_1)ds\\
&+\Bigg(4\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\Bigg)^{n-2}16L^2T_1\int\limits_0^{T_1}\big(16L^2T_1(T_1-s)\big)\mathbf{E}\|\phi(0,\cdot)\|^2_{L_2(\mathbb{R}^d)}ds\\
&+\Bigg(4\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\Bigg)^{n-3}16L^2T_1\int\limits_0^{T_1}\frac{\big(16L^2T_1(T_1-s)\big)^2}{2}\mathbf{E}\|\phi(0,\cdot)\|^2_{L_2(\mathbb{R}^d)}ds+\\
&\ldots+\Bigg(4\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\Bigg)^2 16L^2T_1\int\limits_0^{T_1}\frac{\big(16L^2T_1(T_1-s)\big)^{n-3}}{(n-3)!}\mathbf{E}\|\phi(0,\cdot)\|^2_{L_2(\mathbb{R}^d)}ds\\
&+\Bigg(4\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\Bigg)16L^2T_1\int\limits_0^{T_1}\frac{\big(16L^2T_1(T_1-s)\big)^{n-2}}{(n-2)!}\mathbf{E}\|\phi(0,\cdot)\|^2_{L_2(\mathbb{R}^d)}ds\\
&+16L^2T_1\int\limits_0^{T_1}\frac{\big(16L^2T_1(T_1-s)\big)^{n-1}}{(n-1)!}\mathbf{E}\|\phi(0,\cdot)\|^2_{L_2(\mathbb{R}^d)}ds, \qquad (13)
\end{aligned}
$$

where $C(T_1) = S(T_1) + \left(4 \sup\limits_{0\leqslant t\leqslant T} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}$. It is easy to see that if T_1 is small enough and assumption (3) is true, then the the right-hand of (13) is not more than

$$\frac{S(T_1)}{1 - 4\sup\limits_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx} + \frac{16L^2T_1\cdot S(T_1)\cdot\int\limits_0^{T_1}\exp\{16L^2T_1(T_1-s)\}ds}{1 - 4\sup\limits_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx}$$

$$+\frac{C(T_1)\cdot\exp\{16L^2T_1^2\}}{1 - 4\sup\limits_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx} + \frac{16L^2T_1\cdot\int\limits_0^{T_1}\exp\{16L^2T_1(T_1-s)\}ds}{1 - 4\sup\limits_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx}\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}$$

$$= \frac{\big(S(T_1)+C(T_1)\big)\cdot\exp\{16L^2T_1^2\} + \big(\exp\{16L^2T_1^2\}-1\big)\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}}{1 - 4\sup\limits_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx} > 0.$$

Thus there exists $c(T_1) > 0$ such that for an arbitrary $n \in \{1, 2, \ldots\}$

$$\sup_{0\leqslant t\leqslant T_1}\mathbf{E}\|u^{(n)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} \leqslant c(T_1). \tag{14}$$

1.2. Second let us prove that $\big(u^{(n)}(t,\,\cdot\,), n \in \{1, 2, \ldots\}\big)$, $0 < t \leqslant T_1$, is convergent. In order to do it we estimate $\sup\limits_{0\leqslant t\leqslant T_1}\mathbf{E}\|u^{(n+1)}(t,\,\cdot\,) - u^{(n)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}$, $n \in \{0, 1, \ldots\}$.

If $n = 0$, then we obtain, taking into account estimate (14),

$$\sup_{0\leqslant t\leqslant T_1}\mathbf{E}\|u^{(1)}(t,\,\cdot\,) - u^{(0)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} \leqslant 2\sup_{0\leqslant t\leqslant T_1}\mathbf{E}\|u^{(1)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}$$
$$+ 2\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} < \infty.\text{—}$$

If $n \in \{1, 2, \ldots\}$, then we obtain, taking into account estimates from **1.1**,

$$\sup_{0\leqslant t\leqslant T_1}\mathbf{E}\|u^{(n+1)}(t,\,\cdot\,) - u^{(n)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} \leqslant 2\left(\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)$$

$$\times\sup_{0\leqslant t\leqslant T_1}\mathbf{E}\|u^{(n-1)}(t,\,\cdot\,) - u^{(n)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} + 2L^2T_1\int\limits_0^{T_1}\mathbf{E}\|u^{(n-1)}(s,\,\cdot\,) - u^{(n)}(s,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}ds$$

$$\leqslant \left(2\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx + 2L^2T_1^2\right)\sup_{0\leqslant t\leqslant T_1}\mathbf{E}\|u^{(n-1)}(t,\,\cdot\,) - u^{(n)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} \leqslant \ldots$$

$$\leqslant \left(2\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx + 2L^2T_1^2\right)^n\sup_{0\leqslant t\leqslant T_1}\mathbf{E}\|u^{(0)}(t,\,\cdot\,) - u^{(1)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}.$$

Due to assumption (3) and choose of small T_1, $\sup\limits_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx + L^2T_1^2 < \frac{1}{2}$, therefore $\left(2\sup\limits_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx+2L^2T_1^2\right)^n<1$ and we conclude

$$\lim_{m,n\to\infty}\sup_{0\leqslant t\leqslant T_1}\sqrt{\mathbf{E}\|u^{(n)}(t,\,\cdot\,)-u^{(m)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}}=$$
$$=\lim_{m,n\to\infty}\sup_{0\leqslant t\leqslant T_1}\sqrt{\mathbf{E}\left\|\sum_{i=m-1}^{n-1}\big(u^{(i+1)}(t,\,\cdot\,)-u^{(i)}(t,\,\cdot\,)\big)\right\|^2_{L_2(\mathbb{R}^d)}}\leqslant$$
$$\leqslant\lim_{m,n\to\infty}\sum_{i=m-1}^{n-1}\sqrt{\sup_{0\leqslant t\leqslant T_1}\mathbf{E}\|u^{(i+1)}(t,\,\cdot\,)-u^{(i)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}}\leqslant$$
$$\leqslant\sqrt{\sup_{0\leqslant t\leqslant T_1}\mathbf{E}\|u^{(1)}(t,\,\cdot\,)-u^{(0)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}}\times$$
$$\times\lim_{m,n\to\infty}\sum_{i=m-1}^{n-1}\sqrt{\left(2\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx+2L^2T_1^2\right)^i}=0.$$

Thus, $\big(u^{(n)}(t,\,\cdot\,), n\in\{1,2,\ldots\}\big)$, $0<t\leqslant T_1$, is a Cauchy sequence. Consequently, there is a limiting function $u(t,\,\cdot\,)\in L_2(\mathbb{R}^d)$, $0<t\leqslant T_1$, such that

$$\lim_{n\to\infty}\sup_{0\leqslant t\leqslant T_1}\mathbf{E}\|u^{(n)}(t,\,\cdot\,)-u(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}=0. \tag{15}$$

From (14), it follows from Fatou's Lemma that

$$\sup_{0\leqslant t\leqslant T_1}\mathbf{E}\|u(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}\leqslant c(T_1).$$

The function u is $\mathcal{F}_t$-measurable as a limit of $\mathcal{F}_t$-measurable functions.

1.3. Next we show that $u(t,\,\cdot\,)$, $0<t\leqslant T_1$, solves the equation (6). To this end, we need to pass to the limit in the identity (8). Taking into account (15), we have

$$\lim_{n\to\infty}\sup_{0\leqslant t\leqslant T_1}\mathbf{E}\left\|\int\limits_{\mathbb{R}^d}\big(b(t,\,\cdot\,,u^{(n-1)}(\alpha(t),\xi),\xi)-b(t,\,\cdot\,,u(\alpha(t),\xi),\xi)\big)d\xi\right\|^2_{L_2(\mathbb{R}^d)}\leqslant$$
$$\leqslant\left(\sup_{0\leqslant t\leqslant T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx\right)\lim_{n\to\infty}\sup_{0\leqslant t\leqslant T_1}\mathbf{E}\|u^{(n-1)}(t,\,\cdot\,)-u(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}=0,$$

$$\lim_{n\to\infty}\sup_{0\leqslant t\leqslant T_1}\mathbf{E}\left\|\int\limits_0^t\big(f(s,u^{(n-1)}(\alpha(s),\,\cdot\,),\,\cdot\,)-f(s,u(\alpha(s),\,\cdot\,),\,\cdot\,)\big)ds\right\|^2_{L_2(\mathbb{R}^d)}\leqslant$$
$$\leqslant L^2T_1\lim_{n\to\infty}\int\limits_{-r}^{\alpha(T_1)}\mathbf{E}\|u^{(n-1)}(s,\,\cdot\,)-u(s,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}ds\leqslant$$
$$\leqslant L^2T_1^2\lim_{n\to\infty}\sup_{0\leqslant t\leqslant T_1}\mathbf{E}\|u^{(n-1)}(t,\,\cdot\,)-u(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}=0.$$

Therefore, passing to the limit in (8), we have

$$u(t,\cdot) = \phi(0,\,\cdot\,) + \int\limits_{\mathbb{R}^d} b(0,\,\cdot\,,\phi(-r,\xi),\xi)d\xi - \int\limits_{\mathbb{R}^d} b(t,\,\cdot\,,u(\alpha(t),\xi),\xi)d\xi$$
$$+ \int\limits_0^t f(s,u(\alpha(s),\,\cdot\,),\,\cdot\,)ds + \int\limits_0^t \sigma(s,\cdot)d\beta(s),\ 0 < t \leqslant T_1,$$

— the solution to (6) on $[0, T_1]$. This procedure can be repeated in order to extend the solution to the entire interval $[0, T]$ in finitely many steps, thereby completing the proof.

2. Proof of the theorem 2. Let prove the desired result under the hypothesis *M1*. From now on x is supposed to be fixed.

2.1. Let u_2 solve the problem

$$d\left(u_2(t,\,\cdot\,) + \int\limits_{\mathbb{R}^d} b_2(t,\,\cdot\,,u_2(\alpha(t),\xi),\xi)d\xi\right) = f_2(t,u_2(\alpha(t),\,\cdot\,),\,\cdot\,)dt + \sigma(t,\,\cdot\,)d\beta(t),\ 0 < t \leqslant T,$$
$$u_2(t,\,\cdot\,) = \phi_2(t,\,\cdot\,),\ \ -r \leqslant t \leqslant 0,$$

i.e. satisfy the following identities

$$\left(u_2(t,\,\cdot\,) + \int\limits_{\mathbb{R}^d} b_2(t,\,\cdot\,,u_2(\alpha(t),\xi),\xi)d\xi\right) - \left(u_2(0,\,\cdot\,) + \int\limits_{\mathbb{R}^d} b_2(0,\,\cdot\,,u_2(\alpha(0),\xi),\xi)d\xi\right)$$
$$= \int\limits_0^t f_2(s,u_2(\alpha(s),\,\cdot\,),\,\cdot\,)ds + \int\limits_0^t \sigma(s,\,\cdot\,)d\beta(s),\ 0 < t \leqslant T, \tag{16}$$

$$u_2(t,\,\cdot\,) = \phi_2(t,\,\cdot\,),\ \ -r \leqslant t \leqslant 0. \tag{15*}$$

Let u_3 solve the problem

$$d\left(u_3(t,\,\cdot\,) + \int\limits_{\mathbb{R}^d} b_1(t,\,\cdot\,,u_2(\alpha(t),\xi),\xi)d\xi\right) = f_1(t,u_2(\alpha(t),\,\cdot\,),\,\cdot\,)dt + \sigma(t,\,\cdot\,)d\beta(t),\ 0 < t \leqslant T,$$
$$u_3(t,\,\cdot\,) = \phi_1(t,\,\cdot\,),\ \ -r \leqslant t \leqslant 0,$$

i.e. satisfy the following identities

$$\left(u_3(t,\,\cdot\,) + \int\limits_{\mathbb{R}^d} b_1(t,\,\cdot\,,u_2(\alpha(t),\xi),\xi)d\xi\right) - \left(u_3(0,\,\cdot\,) + \int\limits_{\mathbb{R}^d} b_1(0,\,\cdot\,,u_2(\alpha(0),\xi),\xi)d\xi\right)$$
$$= \int\limits_0^t f_1(s,u_2(\alpha(s),\,\cdot\,),\,\cdot\,)ds + \int\limits_0^t \sigma(s,\,\cdot\,)d\beta(s),\ 0 < t \leqslant T, \tag{17}$$

$$u_3(t,\,\cdot\,)=\phi_1(t,\,\cdot\,),\ \ -r\leqslant t\leqslant 0. \tag{16*}$$

Subtracting (17) – (16*) from (16) – (15*), we obtain

$$\big(u_2(t,\,\cdot\,)-u_3(t,\,\cdot\,)\big)+\underbrace{\int\limits_{\mathbb{R}^d}\big(b_2(t,\,\cdot\,,u_2(\alpha(t),\xi),\xi)d\xi-b_1(t,\,\cdot\,,u_2(\alpha(t),\xi),\xi)\big)d\xi}_{\geqslant 0}$$
$$+\underbrace{\big(u_3(0,\,\cdot\,)-u_2(0,\,\cdot\,)\big)}_{\geqslant 0}+\underbrace{\int\limits_{\mathbb{R}^d}\big(b_1(0,\,\cdot\,,u_2(\alpha(0),\xi),\xi)d\xi-b_2(0,\,\cdot\,,u_2(\alpha(0),\xi),\xi)\big)d\xi}_{=0}$$
$$=\underbrace{\int\limits_0^t\big(f_2(s,u_2(\alpha(s),\,\cdot\,),\,\cdot\,)-f_1(s,u_2(\alpha(s),\,\cdot\,),\,\cdot\,)\big)ds}_{\leqslant 0},\ 0<t\leqslant T,$$

$$u_2(t,\,\cdot\,)-u_3(t,\,\cdot\,)=\underbrace{\phi_2(t,\,\cdot\,)-\phi_1(t,\,\cdot\,)}_{\leqslant 0},\ \ -r\leqslant t\leqslant 0,$$

therefore $u_2\leqslant u_3$ with probability one.

Now let consider u_4 — a solution to

$$d\left(u_4(t,\,\cdot\,)+\int\limits_{\mathbb{R}^d}b_1(t,\,\cdot\,,u_3(\alpha(t),\xi),\xi)d\xi\right)=f_1(t,u_3(\alpha(t),\,\cdot\,),\,\cdot\,)dt+\sigma(t,\,\cdot\,)d\beta(t),\ 0<t\leqslant T,$$
$$u_3(t,\,\cdot\,)=\phi_1(t,\,\cdot\,),\ \ -r\leqslant t\leqslant 0,$$

i.e. is defined from

$$\left(u_4(t,\,\cdot\,)+\int\limits_{\mathbb{R}^d}b_1(t,\,\cdot\,,u_3(\alpha(t),\xi),\xi)d\xi\right)-\left(u_4(0,\,\cdot\,)+\int\limits_{\mathbb{R}^d}b_1(0,\,\cdot\,,u_3(\alpha(0),\xi),\xi)d\xi\right)$$
$$=\int\limits_0^t f_1(s,u_3(\alpha(s),\,\cdot\,),\,\cdot\,)ds+\int\limits_0^t\sigma(s,\,\cdot\,)d\beta(s),\ 0<t\leqslant T, \tag{18}$$

$$u_4(t,\,\cdot\,)=\phi_1(t,\,\cdot\,),\ \ -r\leqslant t\leqslant 0. \tag{17*}$$

Subtracting (18) – (17*) from (17) – (16*), we conclude

$$\big(u_3(t,\cdot)-u_4(t,\cdot)\big)+\underbrace{\int\limits_{\mathbb{R}^d}\big(b_1(t,\cdot,u_2(\alpha(t),\xi),\xi)d\xi-b_1(t,\cdot,u_3(\alpha(t),\xi),\xi)\big)d\xi}_{\geqslant 0}$$

$$+\underbrace{\big(u_4(0,\cdot)-u_3(0,\cdot)\big)}_{=0}+\underbrace{\int\limits_{\mathbb{R}^d}\big(b_1(0,\cdot,u_3(\alpha(0),\xi),\xi)d\xi-b_1(0,\cdot,u_2(\alpha(0),\xi),\xi)\big)d\xi}_{=0}$$

$$=\underbrace{\int\limits_0^t\big(f_1(s,u_2(\alpha(s),\cdot),\cdot)-f_1(s,u_3(\alpha(s),\cdot),\cdot)\big)ds}_{\leqslant 0},\ 0<t\leqslant T,$$

$$u_3(t,\cdot)-u_4(t,\cdot)=\phi_1(t,\cdot)-\phi_1(t,\cdot)=0,\ -r\leqslant t\leqslant 0,$$

therefore $u_3\leqslant u_4$ with probability one.

Continuing in a similar way, one obtain a sequence $\big(u_n, n\in\{2,3,\ldots\}\big)$, fulfilling

$$u_2\leqslant u_3\leqslant u_4\leqslant\ldots\leqslant u_n\leqslant\ldots,$$

where u_n, $n\in\{5,6,\ldots\}$, is defined as

$$\left(u_n(t,\cdot)+\int\limits_{\mathbb{R}^d}b_1(t,\cdot,u_{n-1}(\alpha(t),\xi),\xi)d\xi\right)-\left(u_n(0,\cdot)+\int\limits_{\mathbb{R}^d}b_1(0,\cdot,u_{n-1}(\alpha(0),\xi),\xi)d\xi\right)$$

$$=\int\limits_0^t f_1(s,u_{n-1}(\alpha(s),\cdot),\cdot)ds+\int\limits_0^t\sigma(s,\cdot)d\beta(s),\ 0<t\leqslant T, \tag{19}$$

$$u_n(t,\cdot)=\phi_1(t,\cdot),\ -r\leqslant t\leqslant 0. \tag{18*}$$

2.1 Hereafter we argue in a similar way as in the proof of theorem 1. We establish that $\big(u_n, n\in\{2,3,\ldots\}\big)$ is convergent. In order to do it, we prove that

$$\lim_{n\to\infty}\sup_{0\leqslant t\leqslant T}\mathbf{E}\|u_n(t,\cdot)-u_1(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}=0,$$

where u_1 is defined from

$$\left(u_1(t,\cdot)+\int\limits_{\mathbb{R}^d}b_1(t,\cdot,u_1(\alpha(t),\xi),\xi)d\xi\right)-\left(u_1(0,\cdot)+\int\limits_{\mathbb{R}^d}b_1(0,\cdot,u_1(\alpha(0),\xi),\xi)d\xi\right)$$

$$=\int\limits_0^t f_1(s,u_1(\alpha(s),\cdot),\cdot)ds+\int\limits_0^t\sigma(s,\cdot)d\beta(s),\ 0<t\leqslant T, \tag{20}$$

$$u_1(t,\cdot)=\phi_1(t,\cdot),\ -r\leqslant t\leqslant 0. \tag{19*}$$

It follows from the proof of theorem 1 that there exists a constant $c(T) > 0$ such that $\sup\limits_{0\leqslant t\leqslant T} \mathbf{E}\|u_2(t,\cdot)\|^2_{L_2(\mathbb{R}^d)} \leqslant c(T)$ and $\sup\limits_{0\leqslant t\leqslant T} \mathbf{E}\|u_n(t,\cdot)\|^2_{L_2(\mathbb{R}^d)} \leqslant c(T)$ for $n \in \{3, 4, \ldots\}$. The rest of the proof is similar to the case of theorem 1.

Conclusion. In the present paper we discussed a comparison result for solutions to the Cauchy problems for two stochastic differential equations with delay. On this subject number of authors have obtained their comparison results. We dealed with the Cauchy problems for two neutral stochastic integro-differential equations. Except transient- (or drift-) and diffusion-coefficients, our equations include also one integro-differential term. Basic difference of our case from the case of all earlier investigated problems is presence of this term. We intoduced a concept of solutions to our problems and proved the comparison theorem for them. According to our result, under certain assumptions on coefficients of equations under consideration, their solutions depend on the transient-coefficients in a monotone way.

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Цуканова А. О.

Теорема порівняння для стохастичних інтегро-диференціальних рівнянь нейтрального типу в гільбертовому просторі

Резюме

У даній статті розглядається задача порівняння розв'язків задач Коші для двох стохастичних диференціальних рівнянь із запізненням. У цій галузі безліч авторів отримали

свої результати, які стосуються порівняння розв'язків подібних задач. У роботі розглядаються задачі Коші для двох стохастичних інтегро-диференціальних рівнянь нейтрального типу. Окрім коефіцієнта зносу (переносу) і коефіцієнта дифузії, ці рівняння містять також один інтегро-диференціальний член. Наявність цього інтегрального члена є основною відмінністю досліджуваної задачі від усіх раніше досліджуваних задач. Для цих задач вводяться поняття розв'язків, для яких доведено теорему порівняння. Згідно з отриманим результатом, за деяких припущень на коефіцієнти досліджуваних рівнянь, їх розв'язки монотонно залежать від коефіцієнтів переносу.
Ключові слова: стохастичне диференціальне рівняння, теорема порівняння, гільбертів простір.

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Теорема сравнения для стохатических интегро-дифференциальных уравнений нейтрального типа в гильбертовом пространстве

Резюме

В данной статье рассматривается задача сравнения решений задач Коши для двух стохастических дифференциальных уравнений с запаздыванием. В этой области множество авторов получили свои результаты, касающиеся сравнения решений подобных задач. В данной работе рассматриваются задачи Коши для двух стохастических интегро-дифференциальных уравнений нейтрального типа. Помимо коэффициента сноса (переноса) и коэффициента диффузии, рассматриваемые уравнения содержат также один интегро-дифференциальный член. Наличие этого интегрального члена является основным отличием этой задачи ото всех ранее исследованных задач. Для наших задач вводятся понятия решений, для которых доказана теорема сравнения. Согласно полученному результату, при некоторых предположениях на коэффициенты рассматриваемых уравнений, их решения монотонно зависят от коэффициентов переноса.

Ключевые слова: стохастическое дифференциальное уравнение, теорема сравнения, гильбертово пространство .