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ON THE STRUCTURE OF THE FUNDAMENTAL MATRIX OF THE LINEAR HOMOGENEOUS DIFFERENTIAL SYSTEM OF THE SPECIAL KIND

For the linear homogeneous differenrial system, whose coefficients are represented as an absolutely and uniformly convergent Fourier-series with slowly varying coefficients and frequency, the kind of the fundamental matrix are established by the condidtion of the some resonance relations.
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Introduction. In the theory of linear systems of differential equations is well known the Floquet-Lyapunov theorem [1]. The fundamental matrix $X(t)$ of the linear homogeneous system

$$\frac{dx}{dt} = A(t)x, \ t \in \mathbb{R}, \tag{1}$$

where $A(t)$ – is a continuous T-periodic matrix, has a kind:

$$X(t) = F(t) \ e^{tK},$$

where $F(t)$ – is a T-periodic matrix, and K – is a constant matrix.

There exists many analogues of this theorem for the linear systems of different types, for example, for the systems with quasiperiodic coefficients [2], for the countable systems of differential equations [3], for the differential equations in the Banach spaces [4] and other.

The purpose of this paper is to obtain the kind of the fundamental matrix of the linear systems of the differential equations whose coefficients are represented as an absolutely and uniformly convergent Fourier-series with slowly varying coefficients and frequency by the condition of the some resonance relations.

Notation. Let $G(\varepsilon_0) = \{t,\varepsilon : \ 0 < \varepsilon < \varepsilon_0, \ -L\varepsilon^{-1} \leqslant t \leqslant L\varepsilon^{-1}, \ 0 < L < +\infty\}$.

Definition 1. We say, that a function $p(t,\varepsilon)$ belongs to a class $S_0(m;\varepsilon_0)$ $(m \in \mathbb{N} \cup \{0\})$, if

1) $p : G(\varepsilon_0) \to \mathbb{C}$, 2) $p(t,\varepsilon) \in C^m(G(\varepsilon_0))$ with respect t;
3) $d^k p(t,\varepsilon)/dt^k = \varepsilon^k p_k^*(t,\varepsilon)$ $(0 \leqslant k \leqslant m)$,

$$\|p\|_{S_0(m;\varepsilon_0)} \stackrel{def}{=} \sum_{k=0}^{m} \sup_{G(\varepsilon_0)} |p_k^*(t,\varepsilon)| < +\infty.$$

Under the slowly varying function we mean the function of the class $S_0(m;\varepsilon_0)$.

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Definition 2. We say, that a function $f(t,\varepsilon,\theta(t,\varepsilon))$ belongs to a class $F_0(m;\varepsilon_0;\theta)$ $(m \in \mathbb{N} \cup \{0\})$, if this function can be represented as:

$$f(t,\varepsilon,\theta(t,\varepsilon)) = \sum_{n=-\infty}^{\infty} f_n(t,\varepsilon) \exp(in\theta(t,\varepsilon)),$$

and:

1) $f_n(t,\varepsilon) \in S_0(m;\varepsilon_0)$;
2)

$$\|f\|_{F_0(m;\varepsilon_0;\theta)} \stackrel{def}{=} \sum_{n=-\infty}^{\infty} \|f_n\|_{S_0(m;\varepsilon_0)} < +\infty,$$

3) $\theta(t,\varepsilon) = \int\limits_0^t \varphi(\tau,\varepsilon)d\tau$, $\varphi(t,\varepsilon) \in \mathbb{R}^+$, $\varphi(t,\varepsilon) \in S_0(m;\varepsilon_0)$, $\inf\limits_{G(\varepsilon_0)} \varphi(t,\varepsilon) = \varphi_0 > 0$.

State some properties of the functions of the classes $S_0(m;\varepsilon_0)$, $F_0(m;\varepsilon_0;\theta)$ (the proofs are given in [5]). Let $k = \text{const}$, $p,q \in S_0(m;\varepsilon_0)$, $u,v \in F_0(m;\varepsilon_0;\theta)$. Then kp, $p \pm q$, pq belongs to the class $S_0(m;\varepsilon_0)$, ku, $u \pm v$, uv belongs to the class $F_0(m;\varepsilon_0;\theta)$, and

1) $\|kp\|_{S_0(m;\varepsilon_0)} = |k| \cdot \|p\|_{S_0(m;\varepsilon_0)}$.
2) $\|p \pm q\|_{S_0(m;\varepsilon_0)} \leqslant \|p\|_{S_0(m;\varepsilon_0)} + \|q\|_{S_0(m,\varepsilon_0)}$.
3) $\|pq\|_{S_0(m;\varepsilon_0)} \leqslant 2^m \|p\|_{S_0(m;\varepsilon_0)} \|q\|_{S_0(m;\varepsilon_0)}$.
4) $\|ku\|_{F_0(m;\varepsilon_0;\theta)} = |k| \cdot \|u\|_{F_0(m;\varepsilon_0;\theta)}$.
5) $\|u \pm v\|_{F_0(m;\varepsilon_0;\theta)} \leqslant \|u\|_{F_0(m;\varepsilon_0;\theta)} + \|v\|_{F_0(m;\varepsilon_0;\theta)}$.
6) $\|uv\|_{F_0(m;\varepsilon_0;\theta)} \leqslant 2^m \|u\|_{F_0(m;\varepsilon_0;\theta)} \cdot \|v\|_{F_0(m;\varepsilon_0;\theta)}$.

Definition 3. We say, that a vector-function $a(t,\varepsilon) = \text{colon}(a_1(t,\varepsilon), \ldots, a_N(t,\varepsilon))$ belongs to a class $S_1(m;\varepsilon_0)$, if $a_j(t,\varepsilon) \in S_0(m;\varepsilon_0)$ $(j = \overline{1,N})$. We say, that a matrix-function $A(t,\varepsilon) = (a_{jk}(t,\varepsilon))_{j,k=\overline{1,N}}$ belongs to a class $S_2(m;\varepsilon_0)$, if $a_{jk}(t,\varepsilon) \in S_0(m;\varepsilon_0)$ $(j,k = \overline{1,N})$.

We define the norms:

$$\|a(t,\varepsilon)\|_{S_1(m;\varepsilon_0)} = \max_{1 \leqslant j \leqslant N} \|a_j(t,\varepsilon)\|_{S_0(m;\varepsilon_0)},$$

$$\|A(t,\varepsilon)\|_{S_2(m;\varepsilon_0)} = \max_{1 \leqslant j \leqslant N} \sum_{k=1}^{N} \|a_{jk}(t,\varepsilon)\|_{S_0(m;\varepsilon_0)}.$$

Definition 4. We say, that a vector-function $b(t,\varepsilon,\theta) = \text{colon}(b_1(t,\varepsilon,\theta), \ldots, b_N(t,\varepsilon,\theta))$ belongs to a class $F_1(m;\varepsilon_0;\theta)$, if $b_j(t,\varepsilon,\theta) \in F_0(m;\varepsilon_0;\theta)$ $(j = \overline{1,N})$. We say, that a matrix-function $B(t,\varepsilon,\theta) = (b_{jk}(t,\varepsilon,\theta))_{j,k=\overline{1,N}}$ belongs to a class $F_2(m;\varepsilon_0;\theta)$, if $b_{jk}(t,\varepsilon,\theta) \in F_0(m;\varepsilon_0;\theta)$ $(j,k = \overline{1,N})$.

We define the norms:

$$\|b(t,\varepsilon,\theta)\|_{F_1(m;\varepsilon_0;\theta)} = \max_{1 \leqslant j \leqslant N} \|b_j(t,\varepsilon,\theta)\|_{F_0(m;\varepsilon_0;\theta)},$$

$$\|B(t,\varepsilon,\theta)\|_{F_2(m;\varepsilon_0;\theta)} = \max_{1 \leqslant j \leqslant N} \sum_{k=1}^{N} \|b_{jk}(t,\varepsilon,\theta)\|_{F_0(m;\varepsilon_0;\theta)}.$$

Thus, the matrix $B(t,\varepsilon,\theta)$ has a kind:

$$B(t,\varepsilon,\theta) = \sum_{n=-\infty}^{\infty} B_n(t,\varepsilon)\exp(in\theta(t,\varepsilon)),$$

where $B_n(t,\varepsilon) \in S_2(m;\varepsilon_0)$, and

$$\|B(t,\varepsilon,\theta)\|_{F_2(m;\varepsilon_0;\theta)} \leqslant \sum_{n=-\infty}^{\infty} \|B_n(t,\varepsilon)\|_{S_2(m;\varepsilon_0)}.$$

It is easy to obtain, that, if $A,B \in F_2(m;\varepsilon_0;\theta)$, then $AB \in F_2(m;\varepsilon;\theta)$, and

$$\|AB\|_{F_2(m;\varepsilon_0;\theta)} \leqslant 2^m \|A\|_{F_2(m;\varepsilon_0;\theta)} \cdot \|B\|_{F_2(m;\varepsilon_0;\theta)}. \tag{2}$$

For $A(t,\varepsilon,\theta) \in F_2(m;\varepsilon_0;\theta)$ we denote:

$$\Gamma_n[A] = \frac{1}{2\pi}\int_0^{2\pi} A(t,\varepsilon,\theta)\exp(-in\theta)d\theta \quad (n \in \mathbb{Z}).$$

Main Results

1. Statement of the problem. We consider the next system of differential equations:

$$\frac{dx}{dt} = (\Lambda(t,\varepsilon) + \varepsilon P(t,\varepsilon,\theta))x, \tag{3}$$

where $\varepsilon \in (0,\varepsilon_0)$, $\Lambda(t,\varepsilon) = \mathrm{diag}(\lambda_1(t,\varepsilon),\ldots,\lambda_N(t,\varepsilon)) \in S_2(m;\varepsilon_0)$, $P(t,\varepsilon,\theta) \in F_2(m;\varepsilon_0;\theta)$.

We study the problem about the structure of fundamental matrix of the system (3).

2. Auxiliary results. Consider the linear homogeneous system:

$$\frac{dx}{dt} = \varepsilon A(t,\varepsilon)x, \tag{4}$$

where $\varepsilon \in (0,\varepsilon_0)$, $A(t,\varepsilon) = (a_{jk}(t,\varepsilon))_{j,k=\overline{1,N}} \in S_2(m;\varepsilon_0)$. Then there exists the matrizant $X(t,\varepsilon)$ of the system (4).

Lemma 1. *If $X(t,\varepsilon)$ – is the matrizant of the system (4), then $X(t,\varepsilon)$, $X^{-1}(t,\varepsilon)$ belongs to the class $S_2(m;\varepsilon_0)$.*

Proof. The matrix $X(t,\varepsilon)$ – is satisfied to matrix integral equation:

$$X(t,\varepsilon) = E + \varepsilon\int_0^t A(\tau,\varepsilon)X(\tau,\varepsilon)d\tau, \tag{5}$$

where E – the unit matrix of order N.

We used the Euclid norm:

$$\|A(t,\varepsilon)\| = \sqrt{\sum_{j=1}^{N}\sum_{k=1}^{N}|a_{jk}(t,\varepsilon)|^2}.$$

Based on (5) we obtain:

$$\|X(t,\varepsilon)\| \leqslant \sqrt{N} + \varepsilon \left| \int_0^t \|A(\tau,\varepsilon)\| \cdot \|X(\tau,\varepsilon)\| d\tau \right|.$$

By virtue generalized Gronwall-Bellman Lemma [3, pp. 25–27] we have:

$$\|X(t,\varepsilon)\| \leqslant \sqrt{N} \exp\left(\varepsilon \left| \int_0^t \|A(\tau,\varepsilon)\| d\tau \right| \right).$$

Hence

$$\sup_{G(\varepsilon_0)} \|X(t,\varepsilon)\| \leqslant \sqrt{N} \exp\left(L \cdot \sup_{G(\varepsilon_0)} \|A(t,\varepsilon)\| \right). \tag{6}$$

We have:

$$\frac{dX(t,\varepsilon)}{dt} = \varepsilon A(t,\varepsilon) X(t,\varepsilon). \tag{7}$$

Differentiating the identity (7) $(m-1)$ times, we obtain, that $X(t,\varepsilon)$ belongs to the class $S_2(m;\varepsilon_0)$.

Further we have:

$$\frac{d(X^{-1}(t,\varepsilon))}{dt} = -\varepsilon X^{-1}(t,\varepsilon) A(t,\varepsilon), \quad X^{-1}(0,\varepsilon) = E. \tag{8}$$

Then the matrix $X^{-1}(t,\varepsilon)$ is satisfied to integral equation:

$$X^{-1}(t,\varepsilon) = E - \varepsilon \int_0^t X^{-1}(\tau,\varepsilon) A(\tau,\varepsilon) d\tau. \tag{9}$$

Hence

$$\|X^{-1}(t,\varepsilon)\| \leqslant \sqrt{N} + \varepsilon \left| \int_0^t \|X^{-1}(\tau,\varepsilon)\| \cdot \|A(\tau,\varepsilon)\| d\tau \right|,$$

and by virtue the same generalized Gronwall-Bellman Lemma:

$$\|X^{-1}(t,\varepsilon)\| \leqslant \sqrt{N} \exp\left(\varepsilon \left| \int_0^t \|A(\tau,\varepsilon)\| d\tau \right| \right).$$

Differentiating the identity (8) $(m-1)$ times, we obtain, that $X^{-1}(t,\varepsilon)$ belongs to the class $S_2(m;\varepsilon_0)$ also.

Lemma 1 are proved.

Lemma 2. *Let we have the matrix equation*

$$\frac{dX}{dt} = \varepsilon A(t,\varepsilon,\theta), \tag{10}$$

where $\varepsilon \in (0,\varepsilon_0)$, $A(t,\varepsilon,\theta) \in F_2(m;\varepsilon_0;\theta)$. *Then there exists the solution* $X(t,\varepsilon,\theta)$ *of the equation (10), which belongs to the class* $F_2(m;\varepsilon_0;\theta)$, *and there exists* $K \in (0,+\infty)$, *which not depending from* $A(t,\varepsilon,\theta)$, *such, that*

$$\|X(t,\varepsilon,\theta)\|_{F_2(m;\varepsilon_0;\theta)} \leqslant K\|A(t,\varepsilon,\theta)\|_{F_2(m;\varepsilon_0;\theta)}. \tag{11}$$

Proof. We represent the matrix $A(t,\varepsilon,\theta)$ in a form:

$$A(t,\varepsilon,\theta) = \sum_{n=-\infty}^{\infty} A_n(t,\varepsilon)\exp(in\theta),$$

where $A_n(t,\varepsilon) \in S_2(m;\varepsilon_0)$ $(n \in \mathbb{Z})$. We seek the solution of the equation (10) in a form:

$$X = \sum_{n=-\infty}^{\infty} X_n(t,\varepsilon)\exp(in\theta), \tag{12}$$

where $(N \times N)$-matrices X_n $(n \in \mathbb{Z})$ must be defined. Then we have:

$$\frac{dX_n}{dt} = -in\varphi(t,\varepsilon)X_n + \varepsilon A_n(t,\varepsilon), \quad n \in \mathbb{Z}.$$

In case $n = 0$ we have:

$$\frac{dX_0}{dt} = \varepsilon A_0(t,\varepsilon). \tag{13}$$

Consider the next solution of the equation (13):

$$X_0(t,\varepsilon) = \varepsilon\int_0^t A_0(\tau,\varepsilon)d\tau. \tag{14}$$

Obviously, this solution belongs to the class $S_2(m;\varepsilon_0)$, and there exists $K_0 \in (0,+\infty)$ such, that

$$\|X_0(t,\varepsilon)\|_{S_2(m;\varepsilon_0)} \leqslant K_0\|A_0(t,\varepsilon)\|_{S_2(m;\varepsilon_0)}. \tag{15}$$

In case $n \neq 0$ we state:

$$X_n = \varepsilon e^{-in\theta(t,\varepsilon)}\left(C_n(\varepsilon) + \int_0^t A_n(\tau,\varepsilon)e^{in\theta(\tau,\varepsilon)}d\tau\right), \tag{16}$$

where matrices $C_n(\varepsilon)$ are defined by formulas:

$$C_n(\varepsilon) = \sum_{j=0}^{m-1}(-1)^j\frac{D^j(A_n(t,\varepsilon))}{(in)^{j+1}\varphi(t,\varepsilon)}\Big|_{t=0},$$

and operators D^j are defined by the formulas:

$$D(U) = \frac{d}{dt}\left(\frac{U(t,\varepsilon)}{\varphi(t,\varepsilon)}\right), \quad D^j(U) = D(D^{j-1}(U)), \quad D^0(U) = U.$$

We apply to the integral in (16) the m-fold integration by the parts. We obtain:

$$X_n = \varepsilon \sum_{j=0}^{m-1} (-1)^j \frac{D^j(A_n(t,\varepsilon))}{(in)^{j+1}\varphi(t,\varepsilon)} + \varepsilon(-1)^m \frac{e^{-in\theta(t,\varepsilon)}}{(in)^m} \int\limits_0^t D^m(A_n(\tau,\varepsilon))e^{in\theta(\tau,\varepsilon)}d\tau.$$

Hence

$$\frac{dX_n}{dt} = \varepsilon \sum_{j=0}^{m-2} (-1)^j \frac{D^{j+1}(A_n(t,\varepsilon))}{(in)^{j+1}} +$$

$$+\varepsilon(-1)^m \frac{\varphi(t,\varepsilon)e^{-in\theta(t,\varepsilon)}}{(in)^m} \int\limits_0^t D^m(A_n(\tau,\varepsilon))e^{in\theta(\tau,\varepsilon)}d\tau, \tag{17}$$

$$D^k\left(\frac{dX_n}{dt}\right) = \varepsilon \sum_{j=0}^{m-k-1} (-1)^j \frac{D^{j+k+1}(A_n(t,\varepsilon))}{(in)^{j+1}} +$$

$$+\varepsilon(-1)^{m-k-1} \frac{\varphi(t,\varepsilon)e^{-in\theta(t,\varepsilon)}}{(in)^{m-k-1}} \int\limits_0^t D^m(A_n(\tau,\varepsilon))e^{in\theta(\tau,\varepsilon)}d\tau,\ k = \overline{1,m-1}, \tag{18}$$

Since $A_n(t,\varepsilon) \in S_2(m;\varepsilon_0)$, then $D^k(A_n(t,\varepsilon)) = \varepsilon^k V_{nk}(t,\varepsilon)$, where $V_{nk}(t,\varepsilon) \in S_2(m-k;\varepsilon_0)$ $(k = \overline{0,m})$, and

$$\sum_{k=0}^{m} \sum_{n=-\infty}^{\infty} \sup_{G(\varepsilon_0)} \|V_{nk}(t,\varepsilon)\|_{S_2(m-k;\varepsilon_0)} < +\infty. \tag{19}$$

Based on (17), (18), (19) we can state, that $X_n(t,\varepsilon)$ belongs to the class $S_2(m;\varepsilon_0)$ $(n \in \mathbb{Z})$, and

$$\sum_{n=-\infty}^{\infty} \|X_n(t,\varepsilon)\|_{S_2(m;\varepsilon_0)} < +\infty,$$

therefore the matrix-function, which defined the formula (12), belongs to the class $F_2(m;\varepsilon_0;\theta)$, and there exists $K \in (0,+\infty)$, which not depending from $A(t,\varepsilon,\theta)$, such, that holds unequality (11).

Lemma 2 are proved.

3. Principal result.

Theorem 1. *Let the system (3) such, that:*

$$\inf_{G(\varepsilon_0)} |\mathrm{Re}(\lambda_j(t,\varepsilon) - \lambda_k(t,\varepsilon)| \geqslant \gamma > 0 \quad (j \neq k),$$

and $m \geqslant 1$. *Then there exists* $\varepsilon^* \in (0,\varepsilon_0)$ *such, that forall* $\varepsilon \in (0,\varepsilon^*)$ *there exists the fundamental matrix* $X^{(1)}(t,\varepsilon,\theta)$ *of the system (3), whih has a kind*

$$X^{(1)}(t,\varepsilon,\theta) = R^{(1)}(t,\varepsilon,\theta)\exp\left(\int\limits_0^t \Lambda^{(1)}(\tau,\varepsilon)d\tau\right),$$

where $R^{(1)}(t,\varepsilon,\theta) \in F_2(m-1;\varepsilon^*;\theta)$, $\Lambda^{(1)}(t,\varepsilon)$ *– the diagonal matrix, belonging to the class* $S(m-1;\varepsilon^*)$.

This statement is a consequence of the Principal Result of the paper [6].

Theorem 2. *Let the system (3) such, that*

$$\Lambda(t,\varepsilon) = i\varphi(t,\varepsilon)J,$$

where $\varphi(t,\varepsilon)$ *– is the function in the Definition 2,* $J = \mathrm{diag}(n_1,\ldots,n_N)$, $n_j \in \mathbb{Z}$ $(j=\overline{1,N})$, *and* $m \geqslant 1$. *Then there exists* $\varepsilon^{**} \in (0,\varepsilon_0)$ *such, that for all* $\varepsilon \in (0,\varepsilon^{**})$ *there exists fundamental matrix* $X^{(2)}(t,\varepsilon,\theta)$ *of the system (3), which has a kind:*

$$X^{(2)}(t,\varepsilon,\theta(t,\varepsilon)) = \exp(i\theta(t,\varepsilon)J)R^{(2)}(t,\varepsilon,\theta(t,\varepsilon)),$$

where $R^{(2)}(t,\varepsilon,\theta(t,\varepsilon)) \in F_2(m-1;\varepsilon^{**};\theta)$.

Proof. We make a substitution in the system (3):

$$x = \exp(i\theta(t,\varepsilon)J)y, \tag{20}$$

where y – a new unknown N-dimensional vector. We obtain:

$$\frac{dy}{dt} = \varepsilon Q(t,\varepsilon,\theta)y, \tag{21}$$

where $Q(t,\varepsilon,\theta 0 = \exp(-i\theta(t,\varepsilon)J)P(t,\varepsilon,\theta)\exp(i\theta(t,\varepsilon)J)$ belongs to the class $F_2(m;\varepsilon_0;\theta)$.

Now in the system (21) we make a substitution:

$$y = (E + \varepsilon\Phi(t,\varepsilon,\theta))z, \tag{22}$$

where the matrix Φ are defined from the equation:

$$\varphi(t,\varepsilon)\frac{\partial\Phi}{\partial\theta} = Q(t,\varepsilon,\theta) - U(t,\varepsilon), \tag{23}$$

in which $U(t,\varepsilon) = \Gamma_0[Q(t,\varepsilon,\theta)]$. Then

$$\Phi(t,\varepsilon,\theta) = \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{\Gamma_n[Q(t,\varepsilon,\theta)]}{in\varphi(t,\varepsilon)} \exp(in\theta) \in F_2(m;\varepsilon_0;\theta).$$

As a result of the substitution (22) we obtain:

$$\frac{dz}{dt} = \varepsilon(U(t,\varepsilon) + \varepsilon V(t,\varepsilon,\theta))z, \tag{24}$$

where the matrix V are defined from the equation:

$$(E + \varepsilon\Phi(t,\varepsilon,\theta))V = Q(t,\varepsilon,\theta)\Phi(t,\varepsilon,\theta) - \Phi(t,\varepsilon,\theta)U(t,\varepsilon) - \frac{1}{\varepsilon}\frac{\partial\Phi(t,\varepsilon,\theta)}{\partial t}. \tag{25}$$

The matrix $\frac{1}{\varepsilon}\frac{\partial\Phi}{\partial t}$ belongs to the class $F_2(m-1;\varepsilon_0;\theta)$, then there exists $\varepsilon_2 \in (0,\varepsilon_0)$ such, that for all $\varepsilon \in (0,\varepsilon_2)$ the equation (25) are solved with respect V, anf $V(t,\varepsilon,\theta)$ belongs to the class $F_2(m-1;\varepsilon_2;\theta_0)$.

Together with the system (24) we consider a truncated system:

$$\frac{dz^{(0)}}{dt} = \varepsilon U(t,\varepsilon) z^{(0)}. \tag{26}$$

Continuity of the matrix $U(t,\varepsilon)$ with respect t for all $\varepsilon \in (0,\varepsilon_0)$ guarantees the existence of the matrizant $Z^{(0)}(t,\varepsilon)$ of the system (25), and by virtue the Lemma 1 $Z^{(0)}(t,\varepsilon)$, $(Z^{(0)}(t,\varepsilon))^{-1}$ belongs to the class $S_2(m-1;\varepsilon_0)$.

We make in the system (24) the substitution:

$$z = Z^{(0)}(t,\varepsilon)\xi, \tag{27}$$

where ξ – the new unknown vector. We obtain:

$$\frac{d\xi}{dt} = \varepsilon^2 W(t,\varepsilon,\theta)\xi, \tag{28}$$

where $W = (Z^{(0)}(t,\varepsilon))^{-1}V(t,\varepsilon,\theta)Z^{(0)}(t,\varepsilon)) \in F_2(m-1;\varepsilon_2;\theta)$.

Now we show that there exists the substitution

$$\xi = (E + \varepsilon\Psi(t,\varepsilon,\theta))\eta, \tag{29}$$

where $\Psi \in F_2(m-1;\varepsilon_3;\theta)$ $(\varepsilon_3 \in (0,\varepsilon_2))$, which leads the system (28) to the system:

$$\frac{d\eta}{dt} = O\eta, \tag{30}$$

where O – the null $(N \times N)$-matrix. Really, we define the matrix Ψ from the equation:

$$\frac{d\Psi}{dt} = \varepsilon W(t,\varepsilon,\theta) + \varepsilon^2 W(t,\varepsilon,\theta)\Psi. \tag{31}$$

Consider the truncated equation:

$$\frac{d\Psi^{(0)}}{dt} = \varepsilon W(t,\varepsilon,\theta). \tag{32}$$

By virtue Lemma 2 this equation has a solution $\Psi^{(0)}(t,\varepsilon,\theta) \in F_2(m-1;\varepsilon_2;\theta)$.

We construct the process of successive approximations, usef as initial approximation $\Psi^{(0)}(t,\varepsilon,\theta)$, and the subsequent approximations defining as a solutions from the class $F_2(m-1;\varepsilon_2;\theta)$ of the matrix-equations:

$$\frac{d\Psi^{(k+1)}}{dt} = \varepsilon W(t,\varepsilon,\theta) + \varepsilon^2 W(t,\varepsilon,\theta)\Psi^{(k)},\ k = 0,1,2,\dots\ , \tag{33}$$

Each of theese solutions exists by virtue Lemma 2. Then we have:

$$\frac{d(\Psi^{(k+1)} - \Psi^{(k)})}{dt} = \varepsilon^2 W(t,\varepsilon,\theta)(\Psi^{(k)} - \Psi^{(k-1)}),\ k = 1,2,\dots\ , \tag{34}$$

By virtue Lemma 2 and unequality (2) we obtain:

$$\|\Psi^{(k+1)} - \Psi^{(k)}\|_{F_2(m-1;\varepsilon_2;\theta)} \leqslant \varepsilon 2^{m-1} K \|\Psi^{(k)} - \Psi^{(k-1)}\|_{F_2(m-1;\varepsilon_2;\theta)},\ k = 1,2,\dots$$

(K are defined in the Lemma 2), therefore the convergence of the process (33) are guaranteed by the unequality $0 < \varepsilon < \varepsilon_3$, where $\varepsilon_3 2^{m-1} K < 1$. As a result of the process (33) we obtain the solution $\Psi(t,\varepsilon,\theta)$, belongs to the class $F_2(m-1;\varepsilon_3;\theta)$, of the equation (31).

The matrizant of the system (30) is E. Thus, by virtue (20), (22), (27), (29) we obtain, that the fundamental matrix of the system (3) has a kind:

$$X^{(2)}(t,\varepsilon,\theta) = \exp(i\theta(t,\varepsilon)J)(E + \varepsilon\Phi(t,\varepsilon,\theta))Z^{(0)}(t,\varepsilon)(E + \varepsilon\Psi(t,\varepsilon,\theta)),$$

and the Theorem 2 are proved.

Remark 1. *In the sence of the condidtion of the Theorem 2 we say, that we have a resonance case.*

Conclusion. Thus, the kind of the fundamental matrix of the linear homogeneous systems of the differential equations, whose coefficients are represented as an absolutely and uniformly convergent Fourier-series with slowly varying coefficients and frequency, are obtained in some resonance case.

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Про структуру фундаментальної матриці лінійної однорідної диференціальної системи спеціального вигляду

Резюме

Для лінійної однорідної диференціальної системи, коефіцієнти якої зображувані абсолютно та рівномірно збіжними рядами Фур'є з повільно змінними коефіцієнтами та

частотою, встановлено вигляд фундаментальної матриці за умови виконання певних резонансних співвідношень.
Ключові слова: лінійні диференціальні системи, фундаментальна матриця, ряди Фур'є.

Щёголев С. А.
О структуре фундаментальной матрицы линейной однородной дифференциальной системы специального вида

Резюме

Для линейной однородной дифференциальной системы, коэффициенты которой представимы абсолютно и равномерно сходящимися рядами Фурье с медленно меняющимися коэффициентами и частотой, установлен вид фундаментальной матрицы при условии выполнения некоторых резонансных соотношений.
Ключевые слова: линейные дифференциальные системы, фундаментальная матрица, ряды Фурье.