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ALLIANCE IN THREE PERSON GAMES

In the present paper, we suggest a new concept of an optimal solution (that we call "coalitional equilibrium") based on the concepts of Nash and Berge equilibria. We apply the concept of an optimal solution where the outcome of a deviant coalition cannot increase. Then we determine sufficient conditions of existence of a coalitional equilibrium using the Germeier convolution. The convolution transforms the problem of determining a coalitional equilibrium into finding a saddle point of a special antagonistic game that can be effectively constructed based on the mathematical model of the initiall game. As an example of application, we suggest the proof of existence of a coalitional equilibrium in mixed strategies under "regular" mathematical programming limitations: continuity of players' outcome functions and compactness of sets of strategies. This work is intentionally limited to three persons to avoid cumbersome notations and calculations, even though application of the suggested method to games with more than three players is promising for solving problems of creating stable coalitions.

MSC: 91A12 .
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Introduction. In a three-person game, seven coalitions for joint decision making and five coalitional structures (partitions of all players into non-intersecting coalitions) are possible. Over half a century ago, in 1949, twenty-one-years-old PhD student of Princeton University, John Forbes Nash, suggested in his thesis a concept of "optimal solution" for a coalitional structure consisting of one player each that he called "equilibrium" and, following Borel and von Neumann, he proved the existence of such a solution in mixed strategies.

The concept of equilibrium is based on the stability of a situation considered as an optimal solution against deviation of any player (not necessarily only one). Stability lies in that the deviant's outcome cannot increase. This concept of optimality, later called "Nash equilibrium", later found use in economics, sociology, military sciences

[*] **От редакционной коллегии.** 30 апреля исполнилось 80 лет одному из авторов этой статьи, Владиславу Иосифовичу Жуковскому. Несмотря на бег времени и стремительно меняющуюся жизнь, его дружеское отношение к нам остается неизменным вот уже долгие-долгие годы. Трудолюбие Владислава Иосифовича, способность объяснять простыми словами сложные вещи, чувство юмора и несомненный математический талант остаются образцом для нас и вот уже третьего поколения наших коллег и учеников. Члены редакционной коллегии и математики Одесского национального университета имени И. И. Мечникова желают юбиляру здоровья и сил для достижения поставленных целей, которых — если судить по количеству публикаций — остается еще немало.

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and many other areas. In 45 years, in 1994, Nash, in a common effort with White House employees John Harsanyi and Reichard Selten won the Nobel Prize "for fundamental analysis of equillibria in noncooperative game theory". Within 20 years, Nash developed the foundation of the scientific method that played a great role in the development of world economy.

A different case of a coalitional structure in which all players unite to create a single coalition became the subject of study of multicriteria optimization, founded in 1909 by the Italian economist and sociologist Vilfredo Pareto. Here, the idea is again centered on stability: deviation from an optimal solution causes decrease of one or several criteria. The mathematical theory of multicriteria optimization (multiobjective optimization) developed into a separate modern branch of operational research and also found use in engineering and economics.

What about the "intermediate" coalitional structures that contain at least two coalitions with at least one of these including at least two players? How does one formalize the concept of an optimal solution? The present article is dedicated to this question.

Consider a three-person game with its mathematical model defined by the ordered triplet

$$\Gamma_3 = \langle \{1,2,3\}, \{X_i\}_{i=1,2,3}, \{f_i(x)\}_{i=1,2,3} \rangle \,.$$

In Γ_3, $\{1,2,3\}$ is the set of players, each of whom selects his *strategy* $x_i \in X_i \subset \mathbb{R}^{n_i}, i = 1,2,3$, which results in the *situation* $x = (x_1, x_2, x_3) \in X = \prod_{i=1}^{3} X_i \subset \mathbb{R}^n (n = \sum_{i=1}^{3} n_i)$. For a given situation x in X, the *outcome* of each player $i, i = 1,2,3$ is defined by the value of his *outcome functions* $f_i(x)$. The study of conflicts that are mathematically represented by the three-person game Γ_3, is usually conducted from the standard point of view that defines what players' behavior should be considered optimal (rational, reasonable). The main concepts of optimality in mathematical game theory are [1] the intuitive concepts of *profitability, stability, fairnes and justice*. The "dominant" in the non-coalitional (non cooperative) games, the concept of Nash equilibrium [2], [3], the Berge equilibrium [4], the active equilibrium, and the bargaining equilibrium are based on stability. In addition to the mentioned concepts of optimality, there are several other concepts of optimality prevailing in the non-coalitional game theory. In this class of games, each conflict participant (player) usually pursues his own aims; moreover, the players cannot form coalitions with other players for determining their strategies. The counterpart to the described class of games is the cooperative games [5], in which any unions – coalitions – of players for the purpose of pursuing their common interests as well as the possibility of unlimited negotiations between players that result in the selection and application of a common situation; of course, it is implied that "pacta sunt servanda" (agreements must be commited to). The specific concepts of *individual* [[5], p.117] and *collective or group* [[5], p.125] *rationality* are esential for optimality in cooperative game theory. Individual rationality lies in that each player's outcome is not less than his guaranteed outcome that he can "guarantee" by acting independently (applying his maximin strategy). Collective rationality involves a vector maximum solution such as Pareto, weak pareto, Jeoffrion, Borwein, etc. optimal situations obtained when all players form one coalition.

The present article heavily relies on the concept of a *coalitional structure* of a game (partitioning players into pairwise disjoint subsets). For the three-person game Γ_3,

five coalition structures are possible: $\mathfrak{P}_1 = \{\{1\},\{2\},\{3\}\}$, $\mathfrak{P}_2 = \{\{1,2\},\{3\}\}$, $\mathfrak{P}_3 = \{\{1,3\},\{2\}\}$, $\mathfrak{P}_4 = \{\{1\},\{2,3\}\}$,
$\mathfrak{P}_5 = \{1,2,3\}$. Here, $\mathfrak{P}_1$ corresponds to the non-coalitional nature of a game and $\mathfrak{P}_5$ corresponds to the coalitional nature of a game. The mentioned conditions of individual rationality can be formulated for the coalitional structure $\mathfrak{P}_1$. We will use the following notations: $\forall i \in \{1,2,3\}$, $-i = \{\{1,2,3\}\backslash\{i\}\}$, i.e. for $i = 1 \to -i = \{2,3\}$, for $i = 2 \to -i = \{1,3\}$, and, finally, for $i = 3 \to -i = \{1,2\}$.

Then the condition of individual rationality for a situation $x = (x_1, x_2, x_3) \in X$ means that

$$f_i^o = \max_{x_i \in X_i} \min_{x_{-i} \in X_{-i}} f_i(x_i, x_{-i}) = \min_{x_{-i} \in X_{-i}} f_i(x_i^0, x_{-i}) = f_i(x_i^0, x_{-i}^0) \leqslant f_i(x), \quad i = 1,2,3, \tag{1}$$

i.e. the application of the maximin strateges $x_i, i = 1,2,3$ implies the following inequalities:

$$f_i^0 \leqslant f_i(x), \quad i = 1,2,3. \tag{2}$$

We denote by X^0 the set of individually rational situations of the game Γ_3. For the coalitional structure $\mathfrak{P}_5$ of the game Γ_3: within the set of situations $X^0 \subseteq X$ a situation $x^p \in X^0 \subseteq X$ is *Pareto maximal* in the three-criteria problem $\Gamma_{X^0} = \langle X^0, \{f_i(x)\}_{i=1,2,3}\rangle$ if $\forall x \in X^0$ the system of inequalities $f_i(x) \geqslant f_i(x^p), i = 1,2,3$, of which at least one is strict, is incompatible. According to Karlin lemma [[6], p.71], if

$$\sum_{i=1}^{3} f_i(x^p) = \max_{x \in X^0} \sum_{i=1}^{3} f_i(x), \tag{3}$$

then the situation x^p is Pareto maximal in the problem Γ_{X^0}.

Main Results

1. Conditions of Coalitional Rationality We will formalize the conditions of coalitional rationality for the coalitional structures $\mathfrak{P}_2, \mathfrak{P}_3$ and $\mathfrak{P}_4$. For this purpose, we will use the suitable combination of the concepts of Berge and Nash equilibria.

For the coalitional structure $\mathfrak{P}_2$, the coalitional rationality requires the satisfaction of four inequalities:

$$f_1(x_1^*,x_2^*,x_3) \leqslant f_1(x^*) \quad \forall x_3 \in X_3, \tag{4a}$$

$$f_2(x_1^*,x_2^*,x_3) \leqslant f_2(x^*) \quad \forall x_3 \in X_3, \tag{4b}$$

$$f_1(x_1,x_2,x_3^*) \leqslant f_1(x^*) \quad \forall x_j \in X_j \ (j = 1,2), \tag{4c}$$

$$f_2(x_1,x_2,x_3^*) \leqslant f_2(x^*) \quad \forall x_j \in X_j \ (j = 1,2); \tag{4d}$$

for $\mathfrak{P}_3$:

$$f_1(x_1,x_2^*,x_3) \leqslant f_1(x^*) \quad \forall x_k \in X_k \ (k = 1,3), \tag{5a}$$

$$f_3(x_1,x_2^*,x_3) \leqslant f_3(x^*) \quad \forall x_k \in X_k \ (k = 1,3), \tag{5b}$$

$$f_1(x_1^*,x_2,x_3^*) \leqslant f_1(x^*) \quad \forall x_2 \in X_2, \tag{5c}$$

$$f_3(x_1^*,x_2,x_3^*) \leqslant f_3(x^*) \quad \forall x_2 \in X_2; \tag{5d}$$

and, finally, for $\mathfrak{P}_4$:

$$f_2(x_1,x_2^*,x_3^*) \leqslant f_2(x^*) \quad \forall x_1 \in X_1, \tag{6a}$$

$$f_3(x_1,x_2^*,x_3^*) \leqslant f_3(x^*) \quad \forall x_1 \in X_1, \tag{6b}$$

$$f_2(x_1^*,x_2,x_3) \leqslant f_2(x^*) \quad \forall x_l \in X_l \ (l = 2,3), \tag{6c}$$

$$f_3(x_1^*,x_2,x_3) \leqslant f_3(x^*) \quad \forall x_l \in X_l \ (l = 2,3). \tag{6d}$$

A situation $x^* \in X$ that satisfies all the twelve limitations (4a)–(6d) is called *coalitionally rational* for the game Γ_3. The set of coalitionally rationall situations of the game Γ_3 is denoted by X^*; obviously, $X^* \subseteq X$.

In the process of definition of an optimal solution of the game Γ_3, we will use only 6 of the above 13 inequalities (2) and (4a)–(6d), as the other 6 directly follow from the former 6 inequalities.

This reduction in the number of coalitional rationality conditions is justified iby the following two Lemmas.

Lemma 1. *If (4c), (6c), and (6d) are satisfied for a sitaution x^*, then the following statement holds:*

$$f_i(x^*) \geqslant f_i^0 = \max_{x_i} \min_{x_{-i}} f_i(x_i, x_{-i}) = \min_{x_{-i}} f_i(x_i^0, x_{-i}) \quad , i = 1,2,3.$$

where x_i^0 is defined in (1) for $i = 1, 2, 3$.

Proof. Indeed, according to (4c),

$$f_1(x^*) \geqslant f_1(x_1,x_2,x_3^*) \quad \forall x_j \in X_j \ (j = 1,2).$$

When applying first player's strategy $x_1 = x_1^0$ (defined in (1) for $i = 1$), from the previous inequality we get

$$f_1(x^*) \geqslant f_1(x_1^0,x_2,x_3^*) \geqslant \min_{x_2,x_3} f_1(x_1^0,x_2,x_3) = \max_{x_1} \min_{x_2,x_3} f_1(x_1,x_2,x_3) = f_1^0.$$

Analogously, from (6c) follows

$$f_2(x^*) \geqslant f_2(x_1^*,x_2,x_3) \quad \forall x_2 \in X_2,\ x_3 \in X_3;$$

For $x_2 = x_2^0$, (defined in (1) for $i = 2$)

$$f_2(x^*) \geqslant f_2(x_1^*,x_2^0,x_3) \geqslant \min_{x_1,x_3} f_2(x_1,x_2^0,x_3) = \max_{x_2} \min_{x_1,x_3} f_2(x_1,x_2,x_3) = f_2^0.$$

And finally, according to (6d), setting $x_3 = x_3^0$, we get

$$f_3(x^*) \geqslant f_3(x_1^*,x_2,x_3^0) \geqslant \min_{x_1,x_2} f_3(x_1,x_2,x_3^0) = f_3^0.$$

Lemma 2. *The following implications are true:*

$$(5a) \Rightarrow (4a),\ (4c) \Rightarrow (5c),\ (4d) \Rightarrow (6a),\ (6c) \Rightarrow (4b),\ (5b) \Rightarrow (6b),\ (6d) \Rightarrow (5d).$$

Remark 1. *From Lemmas 1 and 2, it immediately follows that it is sufficient to use six inequalities, namely (5a), (4c), (4d), (6c), (5b) and (6d), instead of all 13 inequalities in determining the optimal solution of the game Γ_3.*

Consequently, we arrive to the following concept of the optimal solution of the game Γ_3; from now on, we use the notation $f = (f_1,f_2,f_3) \in \mathbb{R}^3$.

Definition 1. *We will call the pair $(x^*, f(x^*)) \in X \times \mathbb{R}^3$ coalitional equilibrium for the game Γ_3, if the following conditions hold:*

1. The six inequalities:

$$\begin{aligned} \max_{x_1,x_2} f_j(x_1,x_2,x_3^*) &= f_j(x^*) \quad (j = 1,2),\\ \max_{x_1,x_3} f_k(x_1,x_2^*,x_3) &= f_k(x^*) \quad (k = 1,3),\\ \max_{x_2,x_3} f_l(x_1^*,x_2,x_3) &= f_l(x^*) \quad (l = 2,3); \end{aligned} \tag{7}$$

2. The situation $x^* \in X$ is Pareto maximal within the set of coalitionallly rational situations X^* of the game Γ_3, i.e. $\forall x \in X^*$ the system of inequalities $f_i(x) \geqslant f_i(x^*)$ $(i = 1,2,3)$, of which at least one is strict, is incompatible.

Remark 2. *The pair consisting of the situation x^* and corresponding vector of outcomes $f(x^*) = (f_1(x^*),f_2(x^*),f_3(x^*))$, is an appropriate concept of optimal solution for the game Γ_3 as the existence of the pair $(x^*, f(x^*))$ immediately answers the following fundamental questions of the mathematical game theory: a) How the players should behave in the game Γ_3 in terms of strategy selection? and b) what will they "obtain" as a result? Answer: select their strategies x_i^* from the situation $x^* = (x_1^*,x_2^*,x_3^*)$ and the components of the vector $f(x^*) = (f_1(x^*),f_2(x^*),f_3(x^*))$ are the outcomes they get, respectively, after inplementing the situation $x^* = (x_1^*,x_2^*,x_3^*)$*

Remark 3. *We will list the advantages of the suggested coalitional equilibrium solution of the game Γ_3.*

<u>First</u>, according to Lemma 1, the application of x^* ensures the satisfaction of conditions of individual rationality: each player "obtains" an outcome not less than what he can "guarantee" by acting independently using his own maximin strategy.

<u>Second</u>, the situation x^* "leads" all the players to the "greatest" strategies (Pareto maximal relative to other coalitionall rational situations of the game Γ_3). This fact appears to us as an analogue of the collective rationality of the mathematical theory of cooperative games.

<u>Third</u>, satifsation of requirements (4a)–(6d) means that, for example, for the first player, the dual-purpose distribution of his resources, namely, not forgetting about their interests:

first, player 1 aims to provide maximal assistance to the player 2 in the coalition (union) $\{1,2\}$ as a member of the coalition structure $\mathfrak{P}_2$ (requirements (4c) and (4d);

second, player 1 helps player 3 as a member of the coalition $\{1,3\}$ of thecoalition structure $\mathfrak{P}_3$ (requirements (5a) and (5b)). Formalization of these two requirements in the first and second lines of (7) appears to us as a modification of the idea of a Nash equilibrium concept version features two-criteria scoring players; the third line of (7) can already be viewed as a realization of the idea of Berge equilibrium for the same two-criteria case. The second and third players' behavior can be interpreted similarly.

Finally, the property of coalitional rationality is also based on the principle of stability since, thanks to (7), deviation from x^* of any coalition (of one or two players) cannot lead to "increase" of outcomes of the members of the deviant coalition in the game Γ_3 (compared to $f_i(x^*)$ $(i = 1,2,3)$.

Remark 4. *After the optimal solution has been defined, mathematical game theory recommends answering the following two questions:*

1) Does such a solution exist?

2) how does one find it?

The followingl part of the article is dedicated to answering these questions. We will determine sufficient conditions of coalitional equilibrium (section "Sufficient conditions") and prove its existence in mixed strategies under "common" for the game theory limitations (section "Theorem of existence in mixed startegies")

2. Sufficient Conditions We will now proceed to the result that we find "nec (non) plus ultra" (Latin *nothing above that*) of the present article.

We will employ two n-vectors $x = (x_1,x_2,x_3) \in X \subset \mathbb{R}^n$ $(n = \sum_{i=1}^{3} n_i)$ and $z = (z_1,z_2,z_3) \in X$ as well as the following seven seven scalar functions:

$$\begin{aligned}
\varphi_1(x,z) &= f_1(x_1,x_2,x_3^*) - f_1(z),\\
\varphi_2(x,z) &= f_2(x_1,x_2,x_3^*) - f_2(z),\\
\varphi_3(x,z) &= f_1(x_1,x_2^*,x_3) - f_1(z),\\
\varphi_4(x,z) &= f_3(x_1,x_2^*,x_3) - f_3(z),\\
\varphi_5(x,z) &= f_2(x_1^*,x_2,x_3) - f_2(z),\\
\varphi_6(x,z) &= f_3(x_1^*,x_2,x_3) - f_3(z),\\
\varphi_7(x,z) &= \sum_{l=1}^{3} f_l(x) - \sum_{l=1}^{3} f_l(z),
\end{aligned} \tag{8}$$

and using players' outcome functions in the game Γ_3, we introduce the *Germeier convolution* of these seven functions (8)

$$\varphi(x,z) = \max_{k=1,\dots,7} \varphi_k(x,z), \tag{9}$$

defined in $X \times (Z = X) \subset \mathbb{R}^{2n}$, where $X = \prod_{i=1}^{3} X_i$ is the set of situations of the game Γ_3.

A saddle point $(\overline{x}, z^*) \in X \times Z$ of the scalar function $\varphi(x,z)$ (from (8), (9)) in the antagonistic(zero-sum two-person) game

$$\Gamma^\alpha = \langle X, Z = X, \varphi(x,z)\rangle \tag{10}$$

is defined by the chain of inequalities

$$\varphi(x,z^*) \leqslant \varphi(\overline{x},z^*) \leqslant \varphi(\overline{x},z) \quad \forall x \in X,\ z \in X, \tag{11}$$

where $z^* \in X^*$ is the maximin strategy, i.e.

$$\max_{z\in X}\min_{x\in X}\varphi(x,z) = \min_{x\in X}\varphi(x,z^*).$$

Lemma 3. *If in the game Γ^α there is a saddle point $(\overline{x},z^*)$, then the minimax strategy $z^* \in X$ of the game Γ^α is a coalitional equilibrium of the initial game Γ_3.*

Proof By assuming that $z = \overline{x}$ in (11), from (8) we obtain that $\varphi(\overline{x},\overline{x}) = 0$, as all $\varphi_k(\overline{x},\overline{x}) = 0$ $(k = 1,...,7)$. Then, in accordance with (11), (from transitivity) it follows that

$$\begin{aligned}\varphi(x,z^*) = \max\{&f_1(x_1,x_2,z_3^*) - f_1(z^*), f_2(x_1,x_2,z_3^*) - f_2(z^*), f_1(x_1,z_2^*,x_3) - f_1(z^*),\\ &f_3(x_1,z_2^*,x_3) - f_3(z^*), f_2(z_1^*,x_2,x_3) - f_2(z^*), f_3(z_1^*,x_2,x_3) - f_1(z^*),\\ &\sum_{i=1}^{3} f_i(x_1,x_2,x_3) - \sum_{i=1}^{3} f_i(z_1^*,z_2^*,z_3^*)\} \leqslant 0\end{aligned}$$

for $\forall x_i \in X_i$ $(i = 1,2,3)$. This implies the seven following inequalities:

$$\begin{aligned} f_j(x_1,x_2,z_3^*) &\leqslant f_j(z^*) \quad \forall x_j \in X_j\ (j = 1,2),\\ f_k(x_1,z_2^*,x_3) &\leqslant f_k(z^*) \quad \forall x_k \in X_k\ (k = 1,3),\\ f_l(z_1^*,x_2,x_3) &\leqslant f_l(z^*) \quad \forall x_l \in X_l\ (l = 2,3),\\ \sum_{r=1}^{3} f_r(x_1,x_2,x_3) &\leqslant \sum_{r=1}^{3} f_r(z^*) \quad \forall x = (x_1,x_2,x_3) \in X^* \subseteq X. \end{aligned} \tag{12}$$

The first three inequalities in (12) mean that the situation $z^* \in X$ is (because of these inequalities and (7)) coalitionally rational in the game Γ_3. The last inequality in (12) and the inclusion $X^* \subseteq X$ "guarantee" [[6], p. 71] the Pareto maximality of the situation x^* in the three-criteria problem $\Gamma_{X^*} = \langle X^*, \{f_i(x)\}_{i=1,2,3}\rangle$.

Remark 5. *From Lemma 3, we obtain the following constructive method of finding a coalitional equilibrium of the game Γ_3:*

first, build, using (8) and (9), the function $\varphi(x,z)$,

second, find a saddle point $(\overline{x},z^*)$ of the function $\varphi(x,z)$ (satifsfying the chain of inequalities from (11)),

third, find the values of the three functions $f_i(z^*)$ $(i = 1,2,3)$.

Then the pair $(z^*,f(z^*)) = (f_1(z^*),f_2(z^*),f_3(z^*)) \in X \times \mathbb{R}^3$ forms a coalitional equilibrium of the game Γ_3.

In the following section, we will use the following lemma.

Lemma 4. *If $N+1$ scalar functions $\varphi_j(x,z)$ $(j = 1,...,N+1)$ are continuous in $X \times Z$ and the sets $X, Z \in comp(R^n)$ (are compact), then the function*

$$\varphi(x,z) = \max_{j=1,...,N+1} \varphi_j(z,z) \tag{13}$$

is also continuous on $X \times Z$.

The proof of an even more general result can be found in many textbooks on operational research, for example, in [[7], p. 54], it even appeared in textbooks on convex analysis [[8], p. 146].

3. Theorem of Existence in Mixed Strategies

3.1 Mixed Strategy Situations and Mixed Extension of the Game We will present the mixed strategy extension of the game Γ_3 that includes mixed startegy situations and mathematical expectation of the outcome functions.

We will analyze the three-person game Γ_3, assuming continuity of $f_i(x)$ on the product of compacts $X = \prod_{i=1}^{3} X_i$. In each compact $X_i \subset \mathbb{R}^{n_i}$ $(i = 1,2,3)$ we will consider the Borel σ-algebra $\mathcal{B}(X_i)$ – set of subsets of X_i such that $X_i \in \mathcal{B}(X_i)$, where $\mathcal{B}(X_i)$ is continuous relative to the operations of complement and addition of a countable number of sets from $\mathcal{B}(X_i)$; moreover, $\mathcal{B}(X_i)$ is the minimal σ-algebra that contains all completed subsets of the compact X_i.

When there are no situations x^* in the class of pure strategies $x_i \in X_i$ $(i = 1,2,3)$ that satisfy requirements 1 and 2 of Definition 1, following the approach of Borel [9], Von Neumann [10], Nash [3] and their followers, we need to enlarge the set X_i of pure strategies x_i to mixed ones. Then we will establish the existence of the coaltional equilibrium (analog of Definition 1) in the mixed strategy situations game formalized using mixed strategy situations of the game Γ_3.

Thus we will build Borel σ-algebras $\mathcal{B}(X_i)$ based on each compact X_i $(i = 1,2,3)$ and the Borel σ-algebra $\mathcal{B}(X)$ for the set of situations $X = \prod_{i\in\mathbb{N}} X_i$ assuming that $\mathcal{B}(X)$ contains all Cartesian products of Borel σ-algebras $\mathcal{B}(X_i)$ $(i = 1,2,3)$.

According to mathematical game theory, we will associate a *mixed strategy* $\nu_i(\cdot)$ of the player i to a *probability measure in the compact* X_i. By definition [[11], p. 271] and notations from [[12], p. 284], a probablilty measure is a *non-negative* scalar function $\nu_i(\cdot)$ defined on the Borel σ-algebra $\mathcal{B}(X_i)$ of subsets of the compact $X_i \subset \mathbb{R}^n$ satisfying the following two conditions:

1) $\nu_i\left(\bigcup\limits_k Q_k^{(i)}\right) = \bigcup\limits_k \nu_i\left(Q_k^{(i)}\right)$ for any sequence $\{Q_k^{(i)}\}_{k=1}^{\infty}$ of pairwise disjoint elements from $\mathcal{B}(X_i)$ (property of *countable additivity of the function* $\nu_i(\cdot)$);

2) $\nu_i(X_i) = 1$ (property of *normality*) and thus $\nu_i(Q^{(i)}) \leqslant 1$, $\forall Q^{(i)} \in \mathcal{B}(X_i)$.

We will denote the set of mixed strategies of player i $(i = 1,2,3)$ as $\{\nu_i\}$.

We will also note that the product measures $\nu(dx) = \nu_1(dx_1)\nu_2(dx_2)\nu_3(dx_3)$, in accordance with the known definitions from [[11], p. 370] (and notations from [[12], p. 123]) are probability measures in the set of situations X. The set of these probability measures (situations) we will denote by $\{\nu\}$. Note once more that during the process of building of the product measure $\nu(dx)$ as the σ-algebra of the subsets of the set $X_1 \times X_2 \times X_3 = X$, the *minimal* σ-algebra $\mathcal{B}(X)$ containing all Cartesian products $Q^{(1)} \times Q^{(2)} \times Q^{(3)}$, where $Q^{(i)} \in \mathcal{B}(X_i)$ $(i = 1,2,3)$ is selected. From the known properties of probabilistic measures [[14], p. 288; [[11], p. 254] follows that the sets of all possible measures $\nu_i(dx_i)$ $(i = 1,2,3)$ and $\nu(dx)$ are *weakly closed* and *weakly compact in itself* [[11], p. 212, 254; [13], p. 48, 49]. For $\{\nu\}$, for example, it means that for any infinite sequence $\{\nu^{(k)}\}$ $(k = 1,2,...)$ one can select a subsequence $\{\nu^{(k_j)}\}$ $(j = 1,2,...)$ that will *weakly converge* to $\nu^{(0)}(\cdot) \in \{\nu\}$. That is to say, for any continuous in X function $\psi(x)$ the following statement holds:

$$\lim_{j\to\infty}\int_X \psi(x)\nu^{(k_j)}(dx)=\int_X \psi(x)\nu^{(0)}(dx)$$

and $\nu^{(0)}\in\{\nu\}$. Given the continuity of $\psi(x)$, integrals $\int\limits_X \psi(x)\nu(dx)$ (expectations) are defined using the Fubini theorem

$$\lim_{j\to\infty}\int_X \psi(x)\nu(dx)=\int_{X_1}\int_{X_2}\int_{X_3}\psi(x)\nu_3(dx_3)\nu_2(dx_2)\nu_1(dx_1),$$

where the order of integrations can be altered.

Now we introduce the *mixed extension* of the game Γ_3 based on its pure strategies

$$\left\langle \{1,2,3\},\{\nu_i\}_{i=1,2,3},\left\{f_i[\nu]=\int_X f_i[x]\nu(dx)\right\}_{i=1,2,3}\right\rangle, \tag{14}$$

where, as in Γ_3, $\{1,2,3\}$ is the set of players, but $\{\nu_i\}$ is now the set of mixed strategies $\nu_i(\cdot)$ of player i; in the game Γ_3 each player selects his mixed strategy $\nu_i(\cdot)\in\{\nu_i\}$; the expectation (outcome function) of player i is defined on the set of mixed strategy situations $\{\nu\}$ by:

$$f_i(\nu)=\int_X f_i(x)\nu(dx)\quad (i=1,2,3).$$

For the game (14) we will define an analog of the concept of coalitional equilibrium situation X^*.

Definition 2. *A mixed-strategy situation $\nu^*(\cdot)\in\{\nu\}$ is called coalitional equilibrium of the mixed extension (14) (or coalitional equilibrium in mixed strategies for the game Γ_3) if*

first, the situation $\nu^*(\cdot)$ is coalitionally rational for the game (14), i.e.,

$$\begin{aligned}
&\max_{\nu_1(\cdot)\nu_2(\cdot)} f_j(\nu_1,\nu_2,\nu_3^*)=f_j(\nu^*)\ (j=1,2),\\
&\max_{\nu_1(\cdot)\nu_3(\cdot)} f_k(\nu_1,\nu_2^*,\nu_3)=f_k(\nu^*)\ (k=1,3),\\
&\max_{\nu_2(\cdot)\nu_3(\cdot)} f_l(\nu_1^*,\nu_2,\nu_3)=f_l(\nu^*)\ (j=2,3);
\end{aligned} \tag{15}$$

(We will denote the sets of coalitional equilibria of the game (14) by $\{\nu^*\}$);

second, $\nu^*(\cdot)$ is Pareto maximal in the three-criteria problem

$$\langle\{\nu^*\},\{f_i(\nu)\}_{i=1,2,3}\rangle$$

i.e. for all $\nu(\cdot)\in\{\nu^*\}$, the system of inequalities

$$f_i(\nu)\geqslant f_i(\nu^*)\quad (i=1,2,3),$$

of which at least one is strict, is incompatible;

The sufficient condition of Pareto maximality is obvious; it is the essence of the following remark.

Remark 6. *Mixed situation $\nu^*(\cdot) \in \{\nu\}$ is Pareto maximal in $\tilde{\Gamma}_\nu = \langle\{\nu^*\}, \{f_i(\nu)\}_{i=1,2,3}\rangle$ if*

$$\max_{\nu(\cdot)\in\{\nu^*\}} \sum_{i=1}^{3} f_i(\nu) = \sum_{i=1}^{3} f_i(\nu^*).$$

3.2 Preliminaries In this section we provide some prelimary results.

Lemma 5. *Suppose in the game Γ_3 the sets X_i are compact, the outcome functions $f_i(x)$ are continuous on $X = X_1 \times X_2 \times X_3$ and the set of coalitionally equilibrial mixed-strategy situations $\{\nu^*\}$ (satisfying (15)) is not empty.*

Then $\{\nu^*\}$ is weakly compact in itself subset of the set of situations $\{\nu\}$ of the game (14) (in mixed strategies).

Proof. To establish the weak compactness in itself of the set $\{\nu^*\}$, we will select an arbitrary scalar continuous function $\psi(x)$ with domain the compact set X, and an infinite sequence of situations

$$\nu^{(k)}(\cdot) \in \{\nu^*\} \quad (k = 1,2,...) \tag{16}$$

of the game (14) in mixed strategies. From (16) (and therefore from $\{\nu^*\} \subset \{\nu\}$) follows $\{\nu^{(k)}(\cdot)\} \subset \{\nu\}$. As noted above, the set $\{\nu\}$ is weakly compact in itself, therefore the subsequence $\{\nu^{(k_j)}(\cdot)\}$ and the measure $\nu^{(0)}(\cdot) \in \{\nu\}$ such that

$$\lim_{j\to\infty} \int_X \psi(x)\nu^{(k_j)}(dx) = \int_X \psi(x)\nu^{(0)}(dx).$$

exist. We will then apply the regular method of proving such statements (as in, for example, [[15], p. 86]).

Lemma 6. *Compactness (closedness and boundedness) in the criteria space $\mathbb{R}^3$ of the set*

$$f(\{\nu^*\}) = \bigcup_{\nu(\cdot)\in\nu^*} f(\nu),$$

where, as we recall, the vector $f(x) = (f_1(x), f_2(x), f_3(x))$, can be proven analogously.

Lemma 7. *If in game the (14) the sets $X_i \in comp(\mathbb{R}^n)$ and $f_i(\cdot), i = 1,2,3$ are continuous on X, then for the function*

$$\varphi(x,z) = \max_{r=1,\dots,7} \varphi_r(x,z) \tag{17}$$

the following inequality is correct:

$$\max_{r=1,\dots,7} \int_{X\times X} \varphi_r(x,z)\mu(dx)\nu(dz) \leqslant \int_{X\times X} \max_{r=1,\dots,7} \varphi_r(x,z)\mu(dx)\nu(dz) \tag{18}$$

for all $\mu(\cdot) \in \{\nu\}$, $\nu(\cdot) \in \{\nu\}$; here, we recall that the scalar functions $\varphi_r(x,z)$ are defined in (8), (9).

Proof. Indeed, from (17), for all $x,z \in X$, follow the seven inequalities

$$\varphi_r(x,z) \leqslant \max_{j=1,\dots,7} \varphi(x,z) \quad (r = 1,\dots,7).$$

After integration of both parts of these inequalities with an arbitrary product measure $\mu(dx)\nu(dz)$ as the measure being integrated, we obtain

$$\varphi_r(\mu,\nu) = \int_{X\times X} \varphi_r(x,z)\mu(dx)\nu(dz) \leqslant \int_{X\times X} \max \varphi_j(x,z)\mu(dx)\nu(dz)$$

for all $\mu(\cdot) \in \{\nu\}$, $\nu(\cdot) \in \{\nu\}$ and each $r = 1,\dots,7$. Therefore,

$$\max_{r=1,\dots,7} \varphi_r(\mu,\nu) = \max_{r=1,\dots,7} \int_{X\times X} \varphi_r(x,z)\mu(dx)\nu(dz) \leqslant$$
$$\leqslant \int_{X\times X} \max_{r=1,\dots,7} \varphi_r(x,z)\mu(dx)\nu(dz) \quad \forall\mu(\cdot) \in \{\nu\}, \nu(\cdot) \in \{\nu\}$$

which proves (18).

Remark 7. *In fact, (18) is a generalization of the well-known property of the maximization operation: maximum of a sum cannot be greater than the sum of the maximums.*

3.3 Existence Theorem We will provide the main result of this article: the existence of a mixed strategy coalitional equilibrium situation in the game Γ_3 has been proven.

Theorem 1. *If in the game Γ_3 the sets $X_i \in comp(\mathbb{R}^n)$ and $f_i(\cdot)i = \{1,2,3\}$ are continuous on X, then the game has a coalitional equilibriuml mixed-strategy situation.*

Proof. Consider the auxiliary antagonistic game introduced in (10)

$$\Gamma^\alpha = \langle \{1,2\}, \{X, Z = X\}, \varphi(x,z) \rangle .$$

In the game Γ^α, the set X of strategies x of the first player (maximizing $\varphi(x,z)$). A *saddle point* $(\overline{x},z^*) \in X \times X$ of the game Γ^α satisfies, by definition, the following chain of inequalities for all $x \in X$ and $z \in X$

$$\varphi(x,z^*) \leqslant \varphi(\overline{x},z^*) \leqslant \varphi(\overline{x},z).$$

Now we will associate to Γ^α its mixed extension $\tilde{\Gamma}^\alpha = \langle \{1,2\}, \{\mu\}, \{\nu\}, \varphi(\mu,\nu)\rangle$, where $\{\nu\}$ is the set of mixed strategies $\nu(\cdot)$ of the second player, and $\{\mu\} = \{\nu\}$ is the set of mixed strategies $\mu(\cdot)$ of the first player, whose outcome function (expectation) are defined by

$$\varphi(\mu,\nu) = \int_{X\times X} \varphi(x,y)\mu(dx)\nu(dz).$$

A saddle point (μ^0, ν^*) defined by the inequalities

$$\varphi(\mu,\nu^*) \leqslant \varphi(\mu^0,\nu^*) \leqslant \varphi(\mu^0,\nu) \tag{19}$$

for all $\mu(\cdot) \in \{\nu\}$, $\nu(\cdot) \in \{\nu\}$ will also be a solution of the game $\tilde{\Gamma}^\alpha$ (mixed extension of Γ^α).

This pair (μ^0,ν^*) is called a *mixed-strategy solution of* Γ^α.

In 1952, Gliksberg [16] established the theorem of existence of a Nash equilibrium situation in a non-coalitional game of $N \geqslant 2$ persons in mixed strategies, from which we deduce the statement for its particular case – antagonistic game Γ^α: suppose that in the game Γ^α the set $X \subset \mathbb{R}^n$ is non-empty and compact and the outcome function of the first player $\varphi(x,z)$ is continuous in $X \times X$ (we use the continuity of $\varphi(x,z)$ in Lemma 3). Then for the game Γ^α, there exists a solution (μ^0, ν^*) as defined in (19), i.e. there exists a mixed-strategy saddle point.

Given (18), the inequalities (19) takes the following form:

$$\int_{X\times X} \max_{j=1,\dots,7} \varphi_j(x,z)\mu(dx)\nu^*(dz) \leqslant \int_{X\times X} \max_{j=1,\dots,7} \varphi_j(x,z)\mu^0(dx)\nu^*(dz) \leqslant$$
$$\leqslant \int_{X\times X} \max_{j=1,\dots,7} \varphi_j(x,z)\mu^0(dx)\nu(dz)$$

for all $\mu(\cdot) \in \{\nu\}$, $\nu(\cdot) \in \{\nu\}$. Assuming in

$$\varphi(\mu^0,\nu) = \int_{X\times X} \max_{j=1,\dots,7} \varphi_j(x,z)\mu^0(dx)\nu(dz)$$

the measure $\nu_i(dz_i) = \mu_i^0(dx_i)$ ($i \in \mathbb{N}$) and then $\nu(dz) = \mu^0(dx)$. Given (18), we obtain that $\varphi(\mu^0,\mu^0) = 0$. Analogously follows the equality $\varphi(\nu^*,\nu^*) = 0$ and then from (19) we get

$$\varphi(\mu^0,\nu^*) = 0 \tag{20}$$

From $\varphi(\mu^0,\mu^0) = 0$ and the chain of preceding inequalities (using transitivity), we come to

$$\varphi(\mu,\nu^*) = \int_{X\times X} \max_{j=1,\dots,7} \varphi_j(x,z)\mu(dx)\nu^*(dz) \leqslant 0 \quad \forall \mu(\cdot) \in \{\nu\}.$$

In agreement with the Lemma 7, we have

$$0 \geqslant \int_{X\times X} \max_{j=1,\dots,7} \varphi_j(x,z)\mu(dx)\nu^*(dz) \geqslant \max_{j=1,\dots,7} \int_{X\times X} \varphi_j(x,z)\mu(dx)\nu^*(dz)$$

Therefore, for all $j = 1,\dots,7$, we have

$$\int_{X\times X} \varphi_j(x,z)\mu(dx)\nu^*(dz) \leqslant 0 \quad \forall \mu(\cdot) \in \{\nu\}.$$

There are two cases.

First case ($j = 1,\dots,6$). Here, in accordance with (20), (18) and normality of $\nu_j(\cdot)$, we obtain (see (8))

$$0 \geqslant \int_{X\times X} \varphi_j(x,z)\mu(dx)\nu^*(dz) = \int_{X\times X} (f_j(z_1,z_2,z_3) - f_j(z))\mu(dx)\nu^*(dz) =$$
$$= \int_{X\times X} f_j(z_1,z_2,z_3)\mu(dx)\nu^*(dz) - \int_X f_j(z)\nu^*(dz) \int_X \mu(dx) = f_j(\mu_1,\mu_2,\nu_3^*) - f_j(\nu^*)$$
$$\forall \mu_j(\cdot) \in \{\nu_j\} \ (j = 1,2),$$

Analogously,

$$0 \geqslant \int_{X\times X} \varphi_k(x,z)\mu(dx)\nu^*(dz) = f_k(\mu_1,\nu_2^*,\mu_3) - f_k(\nu^*) \quad \forall\mu_k(\cdot) \in \{\nu_k\}\ (k=1,3)$$

$$0 \geqslant \int_{X\times X} \varphi_l(x,z)\mu(dx)\nu^*(dz) = f_l(\nu_1^*,\mu_2,\mu_3) - f_l(\nu^*) \quad \forall\mu_l(\cdot) \in \{\nu_l\}\ (l=2,3).$$

According to Definition 2, $\nu^*(\cdot)$ is a coalitionally rational situation in mixed strategies for the game Γ_3.

Second case ($j=7$) Once again, in accordance with (20), (18) and normality of $\nu(\cdot)$, we obtain

$$0 \geqslant \int_{X\times X} \left[\sum_{r=1}^{7} f_r(x) - \sum_{r=1}^{7} f_r(z)\right] \mu(dx)\nu^*(dz) = \int_X \sum_{r=1}^{7} f_r(x)\mu(dx)\int_X \nu^*(dz) -$$
$$- \int_X \mu(dx)\int_X \sum_{r=1}^{7} f_r(x)\nu^*(dz) = \sum_{r=1}^{7} f_r(\mu) - \sum_{r=1}^{7} f_r(\nu^*) \quad \forall\mu(\cdot)\in\{\nu\}.$$

Then, after considering Remark 7, we see that the mixed-strategy situation $\nu^*(\cdot) \in \{\nu\}$ of the game Γ_3 is Pareto maximal in the problem

$$\tilde{\Gamma}_\nu = \langle \{\nu^*\},\{f_i(\nu)\}_{i=1,2,3}\rangle\,.$$

Therefore, for the mixed-strategy situation $\nu^*(\cdot)$ of the game Γ_3, coalitional rationality as well as Pareto maximality compared to the other coalitionally rational situations have been established. Therefore, from Definition 2, the mixed-strategy situation $\nu^*(\cdot)$ is coalitionally rational in the mixed extension of the game Γ_3 and the pair $(\nu^*,f(\nu^*))$ forms a coalitional equilibrium in mixed strategies for Γ_3.

Conclusion. In this paper we have made the following *new contributions to cooperative games theory* .

First, the concept of coalitional equilibrium (CE) that takes into account interests of any coalition has been introduced.

Second, a practical method of finding CE has been presented, which can be reduced to the determination of a minimax strategy for a special Germeier convolution that can be built using players' outcome functions.

Third, the existence of CE in mixed strategies under "usual" for mathematical programming conditions (continuity of the outcome functions and compactness of the set of strategies) has been proven.

We find that the following *new qualitative results* of the present article are significant as well:

1. the results can be extended to cooperative games of any number of participants (over three);
2. CE "guarantees" the stability of coalitional structures against deviation of any coalitions;
3. CE is applicable, even if the game's coalitional structure change throughout the game or even if the coalitional structures remains unchanged;
4. CE can be used for forming stable unions of players;

and these by far do not exhaust all advantages of CE!

But there is another advantage that we find important to note.

To this day, in the theory of cooperative games, the conditions of individual or collective rationality have been stressed. Individual interests of players are matched by the concept of Nash equilibrium with its "egoistic" character ("to each his own"); mutual support in games is matched by the concept of Berge equilibrium with its "altruism" ("help everyone and forget about your own interests"). However, such "oblivion" is not characteristic for the human nature of the players. This is overcome by the coalitional rationality.

Indeed, in terms of coalitional rationality, player 1, minding his own interests and being a part of the coalition $\{1{,}2\}$ within the coalitional structure $\mathfrak{P}_2$ helps player 2 (element of Berge equilibrium), while being a part of the coalition $\{1{,}3\}$ within the coalitional structure $\mathfrak{P}_3$ supports player 3, but, as we mentioned "not forgetting about himselve". The same statement is valid for the other players. Therefore, coalitional rationality fills the gap between the Nash (NE) and Berge (BE) equilibriums, adding "care about the others" to NE and "care about themselves" to BE.

In this article, the authors see the idea of the Golden rule: one should treat others as one would like others to treat oneself. In the definition of rational equilibrium in the present article the "others" for each players are the members of the coalition the player takes part in.

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Жуковський В. І., Ларбані М.
Альянс в іграх трьох осіб

Резюме

В цій роботі ми пропонуємо нову концепцію оптимального розв'язку (яку ми називаємо «коаліційною рівновагою»), побудовану на ідеях рівноваги за Нешем та за Берже. Ми використовуємо поняття оптимального розв'язку, в якому виграш коаліції, що відхиляється, не може зростати. Після цього за допомогою згортки Гермейера знаходяться достатні умови існування коаліційної рівноваги. Згортка перетворює задачу знаходження коаліційної рівноваги в пошук сідлової точки особливої антагоністичної гри, яка може буди побудована на підставі математичної моделі вихідної гри. В якості прикладу ми даємо доведення існування коаліційної рівноваги в змішаних стратегіях, за «регулярних» обмежень математичного програмування: неперервності функцій виграшу гравців та компактності множин стратегій. Ми обмежуємось випадком гри трьох осіб в цій роботі, щоб уникнути складних позначень та обчислень. Однак застосування запропонованного методу для ігр з більш ніж трьома гравцями може бути багатообіцяючим при розв'язанні задач побудови стійких коалцій.
Ключові слова: максимін, максимум за Парето, макисмум за Слейтером, коаліційна раціональність, результант Гермейера, змішані стратегії .

Жуковский В. И., Ларбани М.
Альянс в играх трех лиц

Резюме

В этой работе мы предлагаем новую концепцию оптимального решения (которую мы называем «коалиционным равновесием»), основанную на идеях равновесия по Нэшу и по Берже. Мы используем понятие оптимального решения, в котором выигрыш отклоняющейся коалиции не может возрастать. Затем, используя свертку Гермейера, находятся достаточные условия существования коалиционного равновесия. Свертка превращает задачу нахождения коалиционного равновесия в поиск седловой точки особой антагонистической игры, которая может быть эффективно построена на основании математической модели исходной игры. В качестве примера мы даем доказательство существования коалиционного равновесия в смешанных стратегиях при «регулярных» ограничениях математического программирования: функции выигрыша игроков предполагаются непрерывными, а множества стратегий компактными. Мы ограничиваемся в этой работе случаем игры трех лиц, чтобы избежать сложных обозначений и вычислений. Однако применение предложенного метода для игр с более чем тремя игроками может быть многообещающим при решении задач построения устойчивых коалиций.
Ключевые слова: максимин, максимум по Парето, максимум по Слейтеру, коалиционная рациональность, результант Гермейера, смешанные стратегии .