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DYNAMICS OF A STOCHASTIC LOTKA–VOLTERRA FOOD CHAIN MODEL

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This paper is concerned with a stochastic Lotka-Volterra food chain model. The existence of the global solution and the ultimate boundedness of moments of the solutions are proved. Moreover, we estimate the average in time of the solution and investigate the extinction and persistence of each species.
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Introduction. Since the 1920's when Vito Volterra employed systems of differential equations to describe the dynamics of a predator-prey population, there has been a large amount of mathematical study on population dynamics. One of the most important type of species interaction in ecology is food chain interaction. The simplest food chain model is the classical Lotka-Volterra predator-prey

$$\begin{cases} \dot{x}(t) = x(t)(a - by(t)), \\ \dot{y}(t) = y(t)(-c + dx(t)), \end{cases} \tag{1}$$

where $x(t)$ and $y(t)$ represent, respectively, the densities of the prey and the predator populations; a, b, c and d are positive constants. Besides, in order to describe better different ecology models, other predator-prey models with various type of functional responses have been investigated in many papers. Meanwhile, there has been considerable interest in food chain models of n pieces, especially models of three species (see [1,2,8,9,11,13,14,18,21,22]). For example, we take a three species Lotka-Volterra food chain model

$$\begin{cases} \dot{x}(t) = x(t)(A - Bx(t) - C_1 y(t)), \\ \dot{y}(t) = y(t)(-D_1 + C_2 x(t) - E_1 z(t)), \\ \dot{z}(t) = z(t)(-D_2 + E_2 y(t)), \end{cases} \tag{2}$$

where $x(t), y(t), z(t)$ are the densities of the lowest-level prey (X), mid-level species (Y), and top predator (Z) at time t, respectively; $A > 0$ in the intrinsic growth rate of X; $B > 0$ is the coefficient of intra specific competition of X; $D_1 > 0$ and $D_2 > 0$ represent the natural death rate of the mid-level predator respectively, $C_1 > 0$ and $E_1 > 0$ represent the effect of predation on the lowest-level prey and the mild-level species; $C_2 > 0$ and E_2 represent the efficiency and propagation rate of Y and Z in the presence of their own preys.

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On another direction, stochastic population models have also been received much attention recently. In fact, stochastic models are more realistic than deterministic on since parameters of models are often perturbed environment noise. In [17], the authors studied the existence and uniqueness of the positive solution of a general stochastic Lotka-Volterra model. Then, the asymptotic behavior of the positive solution was also considered in [17] and [6]. Especially, the more detailed study for stochastic predator-prey models can be found [3, 5, 7, 12, 19, 20], etc. In [10], the author considered the persistence of the following stochastic food web model in which the top predator consumes the lowest-level rather than the mid-level

$$\begin{cases} dx(t) = x(t)(a - bx(t) - c_1 y(t))dt + \sum\limits_{j=1}^{2} \sqrt{\sigma_{1j}} dW_j(t), \\ dy(t) = y(t)(-d_1 + c_2 x(t) - e_1 z(t))dt + \sum\limits_{j=1}^{2} \sqrt{\sigma_{2j}} dW_j(t), \\ dz(t) = z(t)(-d_2 + e_2 y(t))dt + \sum\limits_{j=1}^{2} \sqrt{\sigma_{3j}} dW_j(t), \end{cases} \tag{3}$$

where the σ_{ij} are nonnegative constants and the $W_j,\ j = \overline{1,3}$ are independent scalar Brownian motion processes. In this paper, we consider (2) with suppose the rate of growth of each species perturbed by white noise. So that (2) becomes

$$\begin{cases} dx(t) = x(t)(A - Bx(t) - C_1 y(t))dt + \sigma_1 x(t) dW(t), \\ dy(t) = y(t)(-D_1 + C_2 x(t) - E_1 z(t))dt + \sigma_2 y(t) dW(t), \\ dz(t) = z(t)(-D_2 + E_2 y(t))dt + \sigma_3 z(t) dW(t), \end{cases} \tag{4}$$

where σ_1,σ_2 and σ_3 are real constants. Beside that, [15] discussed a system has a unique positive solution and its pth moment is bounded. They also established conditions that the system is persistent in time average and the system is going to be extinction in probability. It looks more general than our model. How ever, our results are better than them.

The goal of this paper is to prove the existence and uniqueness of positive solution to Equation (4). Then we investigate the extinction and persistence of each species with a slightly condition. A brief description of the organization of this article is as follows. The article is divided into three sections. In section 2, the existence and uniqueness of the global solution are proved and the ultimate boundedness of moments of the solution are given. In section 3, we give conditions for the existence and persistence of each species.

Main Results

1. Global positive solution and moment estimation. Throughout this paper, we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{W(t)\}_{t\geqslant 0}$ be a scalar Brownian motion defined on this probability space. Denote by $\mathbb{R}^3_+$ the set $\{(x,y,z) \in \mathbb{R}^3 : x > 0, y > 0, z > 0\}$. Obviously, the coefficients of Equation (4) are locally Lipschitz continuous but do not satisfy the linear growth condition. However, we have

Theorem 1. *For any given initial value $\big(x(0),y(0),z(0)\big) \in \mathbb{R}^3_+$, there is a unique global solution to Equation* (4) *on $t \geqslant 0$ and the solution will remain in $\mathbb{R}^3_+$ almost surely.*

Proof. Since the coefficient of the Equation (4) are locally Lipschitz continuous, for any given value $(x_0,y_0,z_0) \in \mathbb{R}^3_+$, there is a unique local solution $(x(t),y(t),z(t))$ on $t \in [0;\tau_e]$, where τ_e is the explosion time. The solution is global if $\tau_e = \infty$ a.s.

Putting $\alpha = C_1 C_2^{-1}$ and $\beta = \alpha E_1 E_2^{-1}$. For each $k \in \mathbb{N}$, we define the stopping time

$$\tau_k = \inf\left\{t \geqslant 0, x(t) + \alpha y(t) + \beta z(t) < k^{-1} \text{ or } x(t) + \alpha y(t) + \beta z(t) > k\right\}$$

with convention $\inf \emptyset = \infty$. τ_k is increasing, so we put $\tau_\infty = \lim_{k\to\infty} \tau_k$. Let $V(x,y,z) = (x + \alpha y + \beta z) - \ln(x + \alpha y + \beta z)$. Obviously, $V(x,y,z) \geqslant 0$ for all $x > 0, y > 0, z > 0$. By Itô's formula,

$$\begin{aligned} dV(x(t),y(t),z(t)) &= \mathcal{L}V(x(t),y(t),z(t))dt \\ &\quad + (x(t) + \alpha y(t) + \beta z(t) - 1)\frac{\sigma_1 x(t) + \alpha\sigma_2 y(t) + \beta\sigma_3 z(t)}{x(t) + \alpha y(t) + \beta z(t)}dW(t), \end{aligned}$$

where,

$$\begin{aligned} \mathcal{L}V(x,y,z) &= (Ax - Bx^2 - \alpha D_1 y - \beta D_2 z) - \frac{Ax - Bx^2 - \alpha D_1 y - \beta D_2 z}{x + \alpha y + \beta z} \\ &\quad + \frac{1}{2}\left(\frac{\sigma_1 x + \alpha\sigma_2 y + \beta\sigma_3 z}{x + \alpha y + \beta z}\right)^2. \end{aligned}$$

From the formula of $\mathcal{L}V(x,y,z)$ we have

$$\mathcal{L}V(x,y,z) \leq (A+B)x - Bx^2 + D_1 + D_2 + \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) \ \forall x,y,z > 0.$$

It implies that $K := \sup_{(x,y,z)\in\mathbb{R}^3_+} \mathcal{L}V(x,y,z) < \infty$. Consequently,

$$\begin{aligned} &\mathbb{E}V\big(x(t\wedge\tau_k),y(t\wedge\tau_k),z(t\wedge\tau_k)\big) \\ &\qquad = V\big(x(0),y(0),z(0)\big) + \mathbb{E}\int_0^{t\wedge\tau_k} \mathcal{L}V\big(x(s),y(s),z(s)\big)ds, \\ &\qquad \leqslant V\big(x(0),y(0),z(0)\big) + \mathbb{E}\int_0^{t\wedge\tau_k} K ds = V\big(x(0),y(0),z(0)\big) + K\mathbb{E}(t\wedge\tau_k). \end{aligned} \tag{5}$$

Suppose $\tau_\infty < \infty$ with a positive probability. It implies the existence of two positive constants ϵ and $T > 0$ such that $\mathbb{P}\{\tau_\infty < T\} > 2\epsilon$. Hence, there is $k_0 \in \mathbb{N}$ such that $\mathbb{P}\{\tau_k < T\} > \epsilon$ for any $k > k_0$.

Putting $h_k = (k - \ln k) \wedge (k^{-1} + \ln k)$, then $h_k \to \infty$ as $k \to \infty$ and $V(x(\tau_k), y(\tau_k), z(\tau_k)) \geqslant h_k$. It follows from (5) that,

$$\begin{aligned} h_k\epsilon \leqslant h_k\mathbb{P}\{\tau_k < T\} &\leqslant \mathbb{E}V\big(x(T\wedge\tau_k),y(T\wedge\tau_k),z(T\wedge\tau_k)\big), \\ &\leqslant K\mathbb{E}(T\wedge\tau_k) + V\big(x(0),y(0),z(0)\big), \\ &\leqslant KT + V\big(x(0),y(0),z(0)\big) \ \ \forall\, k > k_0. \end{aligned}$$

Let $k \to \infty$ we get a contradiction. This implies that $\tau_\infty = \infty$ a.s. The proof is complete.

Theorem 2. *Let $\big(x(t),y(t),z(t)\big)$ be a solution to Equation* (4) *with the initial value $\big(x(0),y(0),z(0)\big) \in \mathbb{R}^3_+$. There exist $\theta, K_\theta > 0$ such that*

$$\limsup_{t\to\infty} \mathbb{E}\big([x(t)]^{\theta+1} + [y(t)]^{\theta+1} + [z(t)]^{\theta+1}\big) \leqslant K_\theta.$$

Proof. Let $\theta = \Big(\Big[\frac{\sigma_1^2+\sigma_2^2+\sigma_3^2}{2}+1\Big].\max\Big\{1,\frac{1}{D_1},\frac{1}{D_2}\Big\}\Big)^{-1}$. Consider $V(x,y,z) = (x+\alpha y+\beta z)^{\theta+1}$. By Itô's formula,

$$\begin{aligned}
&de^{(\theta+1)\theta t}V\big((x(t),y(t),z(t)\big)\\
&\qquad= e^{(\theta+1)\theta t}\Big(\mathcal{L}V\big(x(t),y(t),z(t)\big) + (\theta+1)\theta V\big(x(t),y(t),z(t)\big)\Big)dt\\
&+(\theta+1)e^{(\theta+1)\theta t}\Big(x(t)+\alpha y(t)+\beta z(t)\Big)^\theta\big(\sigma_1 x(t)+\alpha\sigma_2 y(t)+\beta\sigma_3 z(t)\big)dW(t),
\end{aligned} \tag{6}$$

where,

$$\begin{aligned}
\mathcal{L}V(x,y,z) = (\theta+1)(x+\alpha y+\beta z)^\theta[&(Ax-Bx^2-\alpha D_1 y-\beta D_2 z)\\
&+\frac{\theta}{2(x+\alpha y+\beta z)}(\sigma_1 x+\alpha\sigma_2 y+\beta\sigma_3 z)^2].
\end{aligned}$$

Since $\alpha,\beta>0$, for all $(x,y,z)\in\mathbb{R}^3_+$, we have

$$\begin{aligned}
&\frac{\theta}{2(x+\alpha y+\beta z)}(\sigma_1 x+\alpha\sigma_2 y+\beta\sigma_3 z)^2 + \theta(x+\alpha y+\beta z)\\
&\qquad\leqslant \theta\Big(\frac{\sigma_1^2+\sigma_2^2+\sigma_3^2}{2}+1\Big)(x+\alpha y+\beta z)\\
&\qquad\leqslant \theta\Big[\frac{\sigma_1^2+\sigma_2^2+\sigma_3^2}{2}+1\Big]\max\Big\{1,\frac{1}{D_1},\frac{1}{D_2}\Big\}\Big(x+\alpha D_1 y+\beta D_2 z\Big)\\
&\qquad= x+\alpha D_1 y+\beta D_2 z.
\end{aligned}$$

Thus, there is a $K_3>0$ such that for all $(x,y,z)\in\mathbb{R}^3_+$

$$\begin{aligned}
&\mathcal{L}V(x,y,z) + (\theta+1)\theta V(x,y,z)\\
&\qquad\leqslant (\theta+1)(x+\alpha y+\beta z)^\theta\big((A+1)x-Bx^2-\alpha D_1 y-\beta D_2 z\big) \leqslant K_3.
\end{aligned} \tag{7}$$

Let τ_k be defined in the proof of Theorem 1. Taking expectations on both sides of (6) and (7), we obtain,

$$\begin{aligned}
&\mathbb{E}e^{(\theta+1)\theta(t\wedge\tau_k)}V\big(x(t\wedge\tau_k),y(t\wedge\tau_k),z(t\wedge\tau_k)\big)\\
&\qquad\leqslant V\big(x(0),y(0),z(0)\big) + \frac{K_3}{(\theta+1)\theta}\mathbb{E}\big(e^{(\theta+1)\theta(t\wedge\tau_k)}-1\big).
\end{aligned}$$

Let $k\to\infty$, we get

$$e^{(\theta+1)\theta t}\mathbb{E}V\big(x(t),y(t),z(t)\big) \leqslant V\big(x(0),y(0),z(0)\big) + \frac{K_3}{(\theta+1)\theta}(e^{(\theta+1)\theta t}-1),$$

equivalently,

$$\mathbb{E}V\big(x(t),y(t),z(t)\big) \leqslant V\big(x(0),y(0),z(0)\big)e^{-(\theta+1)\theta t} + \frac{K_3}{(\theta+1)\theta}(1-e^{-(\theta+1)\theta t}),$$

which implies

$$\limsup_{t\to\infty} \mathbb{E}V\big(x(t),y(t),z(t)\big) \leqslant \frac{K_3}{(\theta+1)\theta} =: K_\theta.$$

The proof is complete.

The following result is a direct corollary of this theorem.

Corollary 1. *Equation* (4) *is a stochastically ultimately bounded in the sense that for any* $\epsilon > 0$*, there is a positive constant* $H = H(\epsilon)$ *such that for any initial value* $\big(x(0),y(0),z(0)\big) \in \mathbb{R}^3_+$*, the solution has the property that*

$$\limsup_{t\to\infty} \mathbb{P}\big\{x(t)+y(t)+z(t) \geqslant H\big\} < \epsilon.$$

We now give a estimation for the growth rate of the prey.

Theorem 3. *Let* $\big(x(t),y(t),z(t)\big)$ *be a solution to Equation* (4) *with the initial value* $\big(x(0),y(0),z(0)\big) \in \mathbb{R}^3_+$*. We have*

$$\limsup_{t\to\infty} \frac{x(t)}{\ln t} \leqslant \frac{\sigma_1^2}{B}.$$

Proof. Let $\eta > 0$. In view of Itô's formula,

$$e^{\eta t}x(t) = x(0) + \int_0^t e^{\eta s}\Big((A+\eta)x(s) - Bx^2(s) - C_1 y(s)\Big)ds + \sigma_1 \int_0^t e^{\eta s}x(s)dW(s). \quad (8)$$

Note that $M(t) = \int_0^t e^{\eta s}x(s)dW(s)$ is a real valued continuous local martingale with quadratic form

$$\big\langle M(t),M(t)\big\rangle = \int_0^t e^{2\eta s}x^2(s)ds.$$

For each $\lambda > 0$, it follows from the exponential martingale inequality (see [16]) that

$$\mathbb{P}\Big\{\sup_{0\leqslant t\leqslant k} M(t) - \lambda e^{-\eta k}\big\langle M(t),M(t)\big\rangle \;<\; \frac{e^{\eta k}}{\lambda}\ln k\Big\} \leqslant \frac{1}{k^2}.$$

By the well known Borel-Cantelli lemma, there exists an $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that for any $\omega \in \Omega_0$, there exists a $k_0 = k_0(\omega) \in \mathbb{N}$ satisfying

$$M(t) - \lambda e^{-\eta k}\big\langle M(t),M(t)\big\rangle \;<\; \frac{e^{\eta k}}{\lambda}\ln k, \;\; \forall\, 0 \;\leqslant t \leqslant k,\; k \geqslant k_0.$$

Since for any $0 \leqslant t \leqslant k$,

$$e^{-\eta k}\big\langle M(t),M(t)\big\rangle \leqslant \int_0^t e^{\eta s}x^2(s)ds.$$

We have

$$M(t) \leqslant \int_0^t \lambda e^{\eta s}x^2(s)ds + \frac{e^{\eta k}}{\lambda}\ln k,\ \forall 0 \leqslant t \leqslant k,\ k \geqslant k_0. \tag{9}$$

For $\lambda < \frac{B}{\sigma_1}$, we put $K_\lambda = \sup_{x\in\mathbb{R}^+}\big\{(A+\eta)x - (B-\sigma_1\lambda)x^2\big\} < \infty$. It follows from (8), (9) that

$$e^{\eta t}x(t) \leqslant x(0) + \frac{K_\lambda}{\eta}(e^{\eta t}-1) + \frac{e^{\eta k}\sigma_1 \ln k}{\lambda}.$$

Obviously, if $k \geqslant k_0$ and $k-1 \leqslant t \leqslant k$, the following inequality holds,

$$\frac{x(t)}{\ln t} \leqslant \frac{e^{-\eta k}}{\ln(k-1)}\Big(x(0) - \frac{K_\lambda}{\eta}\Big) + \frac{K_\lambda}{\eta\ln(k-1)} + \frac{e^{\eta}\sigma_1}{\lambda}\frac{\ln k}{\ln(k-1)}.$$

Let $k \to \infty$, we get $\limsup_{t\to\infty}\frac{x(t)}{\ln t} \leqslant \frac{e^{\eta}\sigma_1}{\lambda}$. Letting $\eta \to 0, \lambda \to \frac{B}{\sigma_1}$, we yield the desired assertion.

2. Extinction and persistence. We now give condition for the existence and persistence of the three species. Let $\widehat{x}(t)$ be the solution with the initial value $\widehat{x}(0) = x(0)$ to equation

$$dx(t) = \widehat{x}(t)\big(A - B\widehat{x}(t)\big)dt + \sigma_1\widehat{x}(t)dW(t). \tag{10}$$

By the comparison theorem for stochastic equation, we have $\widehat{x}(t) \geqslant x(t)\ \forall t \geqslant 0$ a.s.

If $A < \frac{\sigma_1^2}{2}, \lim_{t\to\infty}\widehat{x}(t) = 0$ a.s. (see [19]) which implies that $\lim_{t\to\infty}x(t) = 0$. Moreover,

$$\frac{\ln y(t)}{t} \leqslant \frac{\ln y(0)}{t} - D_1 - \frac{\sigma_2^2}{2} + \frac{1}{t}\int_0^t C_2x(s)ds + \frac{W(t)}{t} \tag{11}$$

so it follows from $\lim_{t\to\infty}\frac{W(t)}{t} = 0$ a.s. that $\limsup_{t\to\infty}\frac{\ln y(t)}{t} \leqslant -D_1 - \frac{\sigma_2^2}{2}$. Similarly we have

$$\limsup_{t\to\infty}\frac{\ln z(t)}{t} \leqslant -D_2 - \frac{\sigma_3^2}{2}.$$

As a result, when $A < \frac{\sigma_1^2}{2}$, all species are extinct.

If $A_1 > \frac{\sigma_1^2}{2}$ it is known that $\ln\widehat{x}(t)$ has a unique stationary distribution with the density $f_*(u) = C\exp\big((2A-\sigma_1^2)u - \frac{2B}{\sigma_1}e^u\big)$. For more details, please see [19]. Moreover, we define

$$\int_{\mathbb{R}} uf_*(u)du = \frac{2B}{2A-\sigma^2} := m.$$

By the ergodic theorem,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \widehat{x}(s)ds = m,\ \text{a.s.} \tag{12}$$

Substituting the inequality $x(t) \leqslant \widehat{x}(t)\ \forall t \geqslant 0$ and (12) into (11) gets that if $C_2m \leqslant D_1 + \frac{\sigma_2^2}{2},\ y(t) \to 0$ as $t \to \infty$ with probability 1. It again implies $\lim_{t\to\infty}z(t) = 0$ a.s. By the same way as in the proof of [4, Theorem 2.1], we claim that $\ln x(t)$ converges weakly to f_*. In the other case, we have

Theorem 4. *Let $\big(x(t),y(t),z(t)\big)$ be a solution to Equation* (4) *with the initial value $\big(x(0),y(0),z(0)\big) \in \mathbb{R}^3_+$. Assume that $A > \frac{\sigma_1^2}{2}$. The following claims hold with probability 1.*

a) $\liminf_{t\to\infty} \frac{\ln y(t)}{t} \leqslant 0$. *Moreover, if* $\ C_2 m > D_1 + \frac{\sigma_2^2}{2}$ *then*

b) $\limsup_{t\to\infty} \frac{1}{t}\int_0^t y(s)ds \geqslant \min\Big\{\frac{D_2+\frac{\sigma_3^2}{2}}{E_2}, \frac{B}{C_1C_2}\big(C_2m - D_1 - \frac{\sigma_2^2}{2}\big)\Big\}$*;*

c) $\limsup_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \geqslant \frac{D_1+\frac{\sigma_2^2}{2}}{C_2}$*;*

d) *If* $\frac{B}{C_1C_2}\big(C_2m - D_1 - \frac{\sigma_2^2}{2}\big) > \frac{D_2+\frac{\sigma_3^2}{2}}{E_2}$ *then* $\limsup_{t\to\infty} z(t) > 0$.

Proof. Assume that there is a $\Omega_1 \subset \Omega$, $\mathbb{P}(\Omega_1) > 0$ and $\liminf_{t\to\infty} \frac{\ln y(t)}{t} > 0$. Hence, for $\omega \in \Omega_1, \lim_{t\to\infty} \int_0^t y(s)ds = \infty$. It follows from Itô's formula for $\ln x(t)$ that

$$\begin{aligned}\limsup_{t\to\infty} \frac{\ln x(t)}{t} &\leqslant A - \frac{\sigma_1^2}{2} - \liminf_{t\to\infty} \frac{B}{t}\int_0^t x(s)ds \\ &\quad - \lim_{t\to\infty} \frac{C_1}{t}\int_0^t y(s)ds + \lim_{t\to\infty} \sigma_1^2 \frac{W(t)}{t} = -\infty \text{ a.s. in } \Omega_1.\end{aligned}$$

Hence $\lim_{t\to\infty} x(t) = 0$ for almost $\omega \in \Omega_1$. Combining with (11), we get

$$\limsup_{t\to\infty} \frac{\ln y(t)}{t} \leqslant -D_1 - \frac{\sigma_1^2}{2} \qquad \text{for almost } \omega \in \Omega_1.$$

This contradiction implies that item a) holds almost surely.

Suppose that there exists a subset $\Omega_2 \subset \Omega$ with $\mathbb{P}(\Omega_2) > 0$ such that

$$\limsup_{t\to\infty} \frac{E_2}{t}\int_0^t y(s)ds < D_2 + \frac{\sigma_3^2}{2}.$$

By Itô's formula

$$\frac{\ln z(t)}{t} - \frac{\ln z(0)}{t} \leqslant -D_2 - \frac{\sigma_3^2}{2} + \frac{1}{t}\int_0^t E_2 y(s)ds + \frac{W(t)}{t}.$$

Since $\frac{W(t)}{t} \to 0$ as $t \to \infty$ with probability 1, then for almost we have

$$\omega \in \Omega_2, \limsup_{t\to\infty} \frac{\ln z(t)}{t} < 0.$$

Applying Itô's formula again, we derive that for almost $\omega \in \Omega_2$,

$$\begin{aligned}\liminf_{t\to\infty} \frac{\ln y(t)}{t} &\geqslant -D_1 - \frac{\sigma_2^2}{2} + \lim_{t\to\infty} \frac{C_2}{t}\int_0^t \widehat{x}(s)ds - \lim_{t\to\infty} \frac{E_1}{t}\int_0^t z(s)ds \\ &\quad - \limsup_{t\to\infty} \frac{C_2}{t}\int_0^t \big(\widehat{x}(s) - x(s)\big)ds \qquad (13) \\ &= C_2m - D_1 - \frac{\sigma_2^2}{2} - C_2 \limsup_{t\to\infty} \frac{1}{t}\int_0^t \big(\widehat{x}(s) - x(s)\big)ds.\end{aligned}$$

On the other hand, employing Itô's formula for $\ln \widehat{x}(t) - \ln x(t)$ yields

$$0 \leqslant \frac{\ln \widehat{x}(t) - \ln x(s)}{t} = -\frac{B}{t}\int_0^t \big(\widehat{x}(s) - x(s)\big)ds + \frac{C_1}{t}\int_0^t y(s)ds \ \text{ a.s.} \tag{14}$$

which results in

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \big(\widehat{x}(s) - x(s)\big)ds \leqslant \frac{C_1}{B} \limsup_{t\to\infty} \frac{1}{t}\int_0^t y(s)ds \text{ a.s.} \tag{15}$$

From (13) and (15) imply that for almost $\omega \in \Omega_2$

$$\liminf_{t\to\infty} \frac{\ln y(t)}{t} \geqslant C_2 m - D_1 - \frac{\sigma_2^2}{2} - \frac{C_1 C_2}{B} \limsup_{t\to\infty} \frac{1}{t}\int_0^t y(s)ds.$$

By item a), we get

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t y(s)ds \geqslant \frac{B}{C_1 C_2}\Big(C_2 m - D_1 - \frac{\sigma_2^2}{2}\Big) \quad \text{a.s. in} \quad \Omega_2.$$

This inequality implies item b). It follows from item b) that

$$\limsup_{t\to\infty} \frac{\ln y(t)}{t} \geq 0 \quad \text{a.s. if} \quad C_2 m - D_1 - \frac{\sigma_2^2}{2} > 0.$$

Using (11) we have

$$0 \leqslant \limsup_{t\to\infty} \frac{\ln y(t)}{t} \leqslant -D_1 - \frac{\sigma_2^2}{2} + \limsup_{t\to\infty} \frac{C_2}{t}\int_0^t x(s)ds \ \text{ a.s.}$$

From this inequality, it is easy to obtain item c). We now prove the final claim. Suppose that there is a $\Omega_3 \subset \Omega$ such that $\mathbb{P}(\Omega_3) > 0$ and that $\lim_{t\to\infty} z(t) = 0 \ \forall \omega \in \Omega_3$. Similar to the proof of item b) we claim that for almost $\omega \in \Omega_3$,

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t y(s)ds \geqslant \frac{B}{C_1 C_2}(C_2 m - D_1 - \frac{\sigma_2^2}{2}).$$

Applying Itô's formula to $\ln z(t)$ we obtain that for almost $\omega \in \Omega_3$,

$$\limsup_{t\to\infty} \frac{\ln z(t)}{t} = -D_2 - \frac{\sigma_3^2}{2} + \limsup_{t\to\infty} \frac{E_2}{t}\int_0^t y(s)ds + \lim_{t\to\infty} \frac{W(t)}{t} > 0.$$

This contradiction completes the proof.

Conclusion. This paper is concerned with a stochastic Lotka-Volterra food chain model. The existence of the global solution and the ultimate boundedness of moments of the solutions are proved. Moreover, we estimate the average in time of the solution and investigate the extinction and persistence of each species.

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Чан Дінь Туонґ
Динаміка стохастичної моделі харчового ланцюга Лотки–Вольтерри

Резюме

Робота присвячена вивченню стохастичної моделі харчового ланцюга типу Лотки–Вольтерри. Доведено існування глобального розв'язку та граничної обмеженості його моментів. Більш того, ми отримуємо оцінку усередненного за часом розв'язку та досліджуємо умови вимирання та виживання обох біологічних видів.
Ключові слова: броунівський рух, харчовий ланцюг, модель Лотки–Вольтерри, модель хижак–жертва, стохастичне диференціальне рівняння .

Чан Динь Туонг
Динамика стохастической модели пищевой цепочки Лотки–Вольтерра

Резюме

Работа посвящена изучению стохастической модели пищевой цепочки Лотке–Вольтерра. Доказано существование глобального решения и предельная ограниченность его моментов. Более того, мы получаем оценку усредненного по времени решения и исследуем условия вымирания и выживания каждого биологического вида.
Ключевые слова: броуновское движение, пищевая цепочка, модель Лотке–Вольтерра, модель хищник–жертва, стохастическое дифференциальное уравнение .