

UDC 511

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DIVISORS OF THE GAUSSIAN INTEGERS IN NORM GROUP E_N^+

The divisor function in norm group E_n^+ is investigated, where the set E_n^+ is a multiplicative subgroup in the multiplicative group of classes of residues modulo p^n over $\mathbb{Z}[i]$. The asymptotic formula is obtained.

MSC: 11N37.

Key words: divisor function, Gaussian integers, asymptotic formula.

INTRODUCTION. Let A, B be two infinite sets of positive numbers. We define generalized function of divisors

$$\tau_{A,B}(n) = \# \{(a,b) \in A \times B \mid ab = n\}, (n \in \mathbb{N}).$$

Usually study a behavior in average the function $\tau_{A,B}(n)$, i. e. construct an asymptotic formula for $\sum_{n \leq x} \tau_{A,B}(n)$. In the case $A = B = \mathbb{N}$ we have the classical Dirichlet problem of divisors. In works of Smith and Subbarao [5], Nowak [3], Varbanec and Zarzycki [6] was investigated the case $A = \mathbb{N}, B = B(b_0, q) := \{b \in \mathbb{N} \mid b \equiv b_0 \pmod{q}\}$. In the sequel came to be consider other sets A and B .

Varbanec and Zarzycki [7], Varbanec [8], Nowak [4] generalized this problem on the case of sets A, B , which define as the sets of all positive integers each of which is norm of integer ideal in finite extension of field \mathbb{Q} .

In the present paper we will consider a generalized function of divisors over the ring of Gaussian integers determined in this way:

for every $w \in \mathbb{Z}[i]$ we put

$$\tau(w; E_n^+) = \sum_{\substack{\delta \mid w \\ \delta \in E_n^+}} 1,$$

where $E_n^+ := \{\alpha \in \mathbb{Z}[i] \mid N(\alpha) \equiv 1 \pmod{p^n}\}, (p \equiv 3 \pmod{4}, p \text{ is prime})$.

The set E_n^+ is a multiplicative subgroup in the multiplicative group of classes of residues modulo p^n over $\mathbb{Z}[i]$.

AUXILIARY ARGUMENTS. Throughout the paper, α, β and γ (also with a subscript) denote Gaussian integers; $N(\alpha), Sp(\alpha)$ are a norm (respectively, a trace) of α , i. e. $N(\alpha) = |\alpha|^2, Sp(\alpha) = 2Re(\alpha)$.

We denote by $G = \mathbb{Z}[i] = \{a + bi \in \mathbb{C} \mid a, b \in \mathbb{Z}, i^2 = -1\}$ and G_{p^n} (respectively, $G_{p^n}^*$) an additive group of residue classes (respectively, a multiplicative group of reduced residue classes) modulo p^n over G .

Let δ_0, δ be the Gaussian rationales (i. e. $\delta_0, \delta \in \mathbb{Q}[i]$) not necessarily integers and let m be a rational integer (i. e. $m \in \mathbb{Z}$). For $Res > 1$ we consider the following series

$$\zeta_m(s; \delta_0, \delta) = \sum_{\substack{w \in G \\ w \neq -\delta_0}} \frac{e^{4mi \arg w}}{N(w + \delta_0)^s} e^{\pi i Sp(\delta w)} \quad (1)$$

The function $\zeta_m(s; \delta_0, \delta)$ accepts an analytic extending on all complex plane and calls the Hecke zeta-function.

Lemma 1. The Hecke zeta-function $\zeta_m(s; \delta_0, \delta)$ has the functional equation

$$\pi^{-s} \Gamma(2|m| + s) \zeta_m(s; \delta_0, \delta) = \pi^{-(1-s)} \Gamma(2|m| + 1 - s) \zeta_m(1 - s; -\delta, \delta_0) e^{-\pi i Sp(\delta_0 \bar{\delta})}$$

(here $\bar{\delta}$ is a complexly-conjugate with δ).

Moreover, $\zeta_m(s; \delta_0, \delta)$ is an entire function if $m \neq 0$ or $m = 0$ and δ is not a Gaussian integer. For $m = 0$ and $\delta \in G$, $\zeta_m(s; \delta_0, \delta)$ is a holomorphic function except at $s = 1$, where it has a simple pole with a residue π .

For the proof in case $\zeta_m(s; 0, 0)$ see [1]. The proof in other cases is similar.

Corollary. $\zeta_0(0; \delta_0, \delta) = 0$ if δ_0 is not a Gaussian integer.

Lemma 2. Let $\alpha \in E_n^+$. Then for every ε , $0 < \varepsilon < \frac{1}{2}$ any $T > 1$ in the rectangle

$$R = \{-\varepsilon \leq \text{Res} \leq 1 + \varepsilon, |\text{Im}s| \leq T\}$$

we have

$$(s-1)^2 \left[\zeta_m(s; 0, 0) \left(\zeta_m\left(s; \frac{\alpha}{p^n}, 0\right) - \sum_{\beta \in B} \frac{e^{4mi \arg\left(\frac{\alpha}{p^n} + \beta\right)}}{\left(N\left(\frac{\alpha}{p^n} + \beta\right)\right)^s} \right) \right] = \quad (2)$$

$$= O\left(\varepsilon^{-2} (t^2 + 1) (t^2 + m^2 + 10)^\theta\right)$$

where $\theta = \frac{(1+\varepsilon)(1+\varepsilon-\sigma)}{1+2\varepsilon}$, $\sigma = \text{Res}$.

From now on B denote the set $\{0, \pm 1, \pm i\}$.

A constant in symbol ‘‘O’’ is absolute.

Proof. This assertion follows at once from estimates for

$$(s-1)^2 \left[\zeta_m(s; 0, 0) \left(\zeta_m\left(s; \frac{\alpha}{p^n}, 0\right) - \sum_{\beta \in B} e^{4mi \arg\left(\frac{\alpha}{p^n} + \beta\right)} \left(N\left(\frac{\alpha}{p^n} + \beta\right)\right)^s \right) \right]$$

on vertical legs of the rectangle R and the Phragmen–Lindelof theorem. ■

Lemma 3. Let $\delta \in \mathbb{Q}(i)$, $N(\delta_0) < 1$. Then $\zeta_0(s; \delta_0, 0)$ has the following in the Laurent expansion

$$\zeta_0(s; \delta_0, 0) = \frac{\pi}{s-1} + a'_0(\delta_0) + a_1(\delta_0)(s-1) + \dots,$$

where

$$a_0(\delta_0) = \begin{cases} \pi\gamma + 4L'(1, \chi_4) & \text{if } \delta_0 \in G, \\ \pi\gamma + 4L'(1, \chi_0) + O\left(\min_{\beta \in B} (N(\delta_0 + \beta))^{-1}\right) & \text{if } \delta_0 \in \mathbb{Q}(i), \delta_0 \notin G; \end{cases} \quad \gamma$$

is the Euler’s constant, χ_4 is non-principal character modulo 4.

Proof. For $\delta_0 = 0$ we have $\zeta_0(s; 0, 0) = 4\xi(s)L(s, \chi_4)$, where $\xi(s)$ is the Riemann zeta-function, $L(s, \chi_4)$ is the Dirichlet zeta-function with non-principal character modulo 4.

Hence,

$$a_0(\delta_0) = \pi\gamma + 4L'(1, \chi_4).$$

Since a residue of $\zeta_0(s; \delta_0, 0)$ does not depend on δ_0 , $\delta_0 \neq 0$, we may write

$$\begin{aligned} a_0(\delta_0) - a_0(0) &= \lim_{s \rightarrow 1 \pm 0} (\zeta_0(s; \delta_0, 0) - \zeta_0(s; 0, 0)) = \\ &= \lim_{s \rightarrow 1 \pm 0} \left\{ \frac{1}{(N(\delta_0))^s} + \sum_{N(\beta)=1} \left(\frac{1}{(N(\delta_0+\beta))^s} - \frac{1}{(N(\beta_0))^s} \right) + \right. \\ &\quad \left. + \sum_{N(\beta) \geq 2} \left(\frac{1}{(N(\delta_0+\beta))^s} - \frac{1}{(N(\beta))^s} \right) \right\} = \\ &= \frac{1}{N(\delta_0)} + \sum_{N(\beta)=1} \frac{1}{N(\delta_0+\beta)} - 4 + \sum_{N(\beta) \geq 2} \frac{N(\beta) - N(\beta+\delta_0)}{N(\beta)N(\beta+\delta_0)} \end{aligned}$$

At last, if we observe that for $N(\beta) \rightarrow \infty$

$$\left| \frac{N(\beta) - N(\beta + \delta_0)}{N(\beta)N(\beta + \delta_0)} \right| \leq \frac{c|\delta_0| \cdot |\beta|}{N(\beta)N(\beta + \delta_0)} = O\left(N(\beta)^{1/2} N(\beta + \delta_0)^{-3/2}\right)$$

we obtain the assertion of Lemma. ■

Corollary.

$$\operatorname{res}_{s=1} \left\{ \zeta_0(s; 0, 0) \zeta_0\left(s; \frac{\alpha}{p^n}, 0\right) \frac{y^s}{s} \right\} = \pi^2 y \log y + c_0(\alpha, p^n) y \quad (3)$$

where

$$\begin{aligned} c_0(\alpha, p^n) &= \pi \left(\sum_{\beta \in B} \frac{1}{N(\frac{\alpha}{p^n} + \beta)} + 2\pi \left(\gamma - \frac{1}{2}\right) + 8L'(1, \chi_4) - 4 + \right. \\ &\quad \left. + O\left(N^{\frac{1}{2}}\left(\frac{\alpha}{p^n}\right)\right) \right) \end{aligned} \quad (4)$$

Lemma 4. For every $\varepsilon > 0$ and $T \rightarrow \infty$ the following assertion has place

$$\begin{aligned} \int_{-T}^T \left| \frac{e^{4mi \arg \gamma}}{N(\gamma)^s} \zeta_m\left(s; \frac{\alpha}{\gamma}, 0\right) - \sum_B e^{4mi \arg(\alpha + \beta\gamma)} (N(\alpha + \beta\gamma))^{-s} \right| ds = \\ = O\left(\frac{(T^2 + m^2)^{\frac{1}{2} + \varepsilon}}{N(\gamma)^{1 - \varepsilon}}\right) \end{aligned}$$

with the O-constant depending only on ε .

Lemma 5. Let a, a_0, b, b_0 be complex-valued functions over the ring of Gaussian integers, $|a_0(w)|, |b_0(w)| \leq 1$. Let

$$A(w) = \sum_{\substack{\delta \in G \\ a(\delta) = w}} a_0(\delta)$$

$$B(w) = \sum_{\substack{\delta \in G \\ b(\delta) = w}} b_0(\delta)$$

and let $A(w), B(w) \ll N(w)^\varepsilon$, $\varepsilon > 0$.
Let us assume for $\alpha, \gamma \in G$ and $\text{Res} > 1$

$$f(w; \alpha, \gamma) = \sum_{\substack{w_1, w_2 = w \\ w_2 \equiv \alpha \pmod{\gamma}}} A(w_1) B(w_2)$$

$$F(s) = \sum_w f(w; \alpha, \gamma) N(w)^{-s}.$$

Then for $c \geq 1 + \varepsilon$, $T > 1$, $1 < N(\gamma) \leq x$ we have

$$\begin{aligned} & \sum_{N(w) \leq x} f(w; \alpha, \gamma) = \\ &= \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left[F(s) - \sum_{\beta \in B} B(\alpha + \beta\gamma) N^{-s}(\alpha + \beta\gamma) \sum_w A(w) N(w)^{-s} \right] \frac{x^s}{s} ds + \\ &+ \sum_{\beta \in B} B(\alpha + \beta\gamma) \sum_{w \in \Omega} A(w) + O \left(x^{c+2\varepsilon} T^{-1} (c-1)^2 \min \left\{ 1, \frac{1/2}{N(\gamma)^{1/2}} \right\} \right) + \\ &+ O \left(\frac{x^{1/2+\varepsilon}}{N^{1/2}(\gamma)} \log T \right) \end{aligned} \tag{5}$$

where

$$B = \{\beta \mid N(\beta) = 0, 1\}, \Omega(\alpha) = \left\{ w(\alpha) \mid N(w) \leq \frac{x}{N(\alpha + \beta\gamma)} \right\}.$$

The proofs of Lemma 4 and 5 are similar to proofs of Lemmas 6 and in [8].

MAIN RESULTS. We denote

$$\tau^{(m)}(w; E_n^+) := \sum_{\substack{\alpha \in E_n^+ \\ \alpha \mid w}} e^{4mi \arg w} = e^{4mi \arg w} \tau^{(0)}(w; E_n^+) = e^{4mi \arg w} \tau(w; E_n^+).$$

First we prove the following statement.

Theorem 1. Let p be a prime rational number, $p \equiv 3 \pmod{4}$. For any positive integer n the asymptotic formula

$$\begin{aligned} \sum_{N(w) \leq x} \tau^{(m)}(w; E_n^+) &= \varepsilon_m \left(\frac{\pi^2}{2} \frac{p+1}{p} \frac{x \log x}{p^n} + \right. \\ &+ \frac{\pi x}{4p^n} \frac{p+1}{p} (b_0(\tilde{\chi}_0) + \gamma + L'(1, \chi_4)) + O \left(x^{1/2+\varepsilon} \log T \right) + \\ &\left. + O \left(\left(\frac{T^2 + m^2}{T^2} \right)^{\frac{1}{2}} p^{-n} x^{\frac{1}{2}} \right) \right) \end{aligned}$$

holds, where the parameter $b_0(\tilde{\chi}_0)$ determine in (??)

$$\varepsilon_m = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases} .$$

a parameter $b_0(\tilde{\chi}_0)$ determine in (9) (see the bellow).

Proof. For $m = 0$ we use Lemma 5 with $a(w) = b(w) = w$, $a_0(w) = 1$

$$b_0(w) = \begin{cases} 1 & \text{if } w \in E_n^+ \\ 0 & \text{else} \end{cases}$$

For $Res > 1$ and $\alpha \in E_n^+$ we have:

$$p^{-2ns} \zeta_0(s; 0, 0) \zeta_0\left(s; \frac{\alpha}{p^n}, 0\right) = \sum_w \tau^*(w; \alpha, E_n^+) N(w)^{-s},$$

where

$$\tau(w; \alpha, E_n^+) = \sum_{\substack{w = w_1 w_2 \\ w_2 \equiv \alpha \pmod{p^n} \\ \alpha \in E_n^+}} 1.$$

Hence, by (5),

$$\begin{aligned} & \sum_{N(w) \leq x} \tau(w; \alpha, E_n^+) = \\ & = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \zeta_0(s; 0, 0) \left[p^{-2ns} \zeta_0\left(s; \frac{\alpha}{p^n}, 0\right) - \sum_{\beta \in B} N(\alpha + \beta p^n)^{-s} \right] \frac{x^s}{s} ds + \\ & + \sum_B \sum_{\Omega(\alpha)} 1 + O\left(\frac{x^{c-2\varepsilon}}{T^{(c-1)^2}} \min\left\{1, \left(\frac{x}{p^{3n}}\right)^{1/2}\right\}\right) + O\left(x^{1/2+\varepsilon} p^{-n} \log T\right). \end{aligned} \quad (6)$$

In order to estimate the integral in (6) we take $c = 1 + 3\varepsilon$ and use the residual theorem. Thus Lemmas 1 and 2 give

$$\begin{aligned} & \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \zeta_0(s; 0, 0) \left[\zeta_0\left(s; \frac{\alpha}{p^n}, 0\right) - \sum_{\beta \in B} N(\alpha + \beta p^n)^{-s} \right] \left(\frac{x}{p^{2n}}\right)^s \frac{ds}{s} = \\ & = \operatorname{res}_{s=1} \left\{ \zeta_0(s) \left[\zeta_0\left(s; \frac{\alpha}{p^n}, 0\right) - \sum_B N\left(\frac{\alpha}{p^n} + \beta\right)^{-s} \right] \left(\frac{x}{p^{2n}}\right)^s \frac{1}{s} \right\} + \\ & + \int_{1/2-iT}^{1/2+iT} \zeta_0(s; 0, 0) \left[\zeta_0\left(s; \frac{\alpha}{p^n}, 0\right) - \sum_{\beta \in B} N(\alpha + \beta p^n)^{-s} \right] \left(\frac{x}{p^{2n}}\right)^s \frac{ds}{s} + \\ & + O(x^{1+3\varepsilon} T^{-1} p^{-2n}) + O\left((xp^{-2n})^{\frac{1}{2}+\varepsilon}\right) \quad (??) \end{aligned}$$

Applying Cauchy inequality in an integral in (??), we obtain

$$\begin{aligned} I & = \left| \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \zeta_0(s; 0, 0) \left[\zeta_0\left(s; \frac{\alpha}{p^n}, 0\right) - \sum_B N\left(\frac{\alpha}{p^n} + \beta\right)^{-s} \right] \left(\frac{x}{p^{2n}}\right)^s \frac{ds}{s} \right| \leq \\ & \leq x^{1/2} \left(\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} |\zeta_0(s; 0, 0)|^2 \frac{ds}{s} \right)^{1/2} \left(\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \left| p^{-2ns} \zeta_0\left(s; \frac{\alpha}{p^n}, 0\right) - \sum_B \frac{1}{N(\alpha + \beta p^n)} \right|^2 \frac{|ds|}{|s|} \right)^{1/2} \end{aligned}$$

Since $\zeta_0(s; 0, 0) = 4\xi(s)L(s, \chi_4)$ we infer

$$\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} |\zeta_0(s; 0, 0)|^2 \frac{|ds|}{|s|} \ll \left(\int_1^T |\xi(s)|^4 \frac{dt}{t} \right)^{\frac{1}{2}} \left(\int_1^T |L(s, \chi_4)|^4 \frac{dt}{t} \right)^{\frac{1}{2}} \ll \log^5 T$$

(we use estimates of mean value for fourth moment of $\xi(s)$ and $L(s, \chi_4)$, see [2]).
Moreover, Lemma 4 gives

$$\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \left| p^{-2ns} \zeta_0\left(s; \frac{\alpha}{p^n}, 0\right) - \sum_{\mathbf{B}} N(\alpha + \beta p^n)^{-s} \right|^2 \frac{|ds|}{|s|} \ll \frac{T^\varepsilon}{p^{2n(1-\varepsilon)}}$$

Therefore,

$$I \ll x^{1/2} p^{-n+\varepsilon} T^\varepsilon \ll x^{1/2+\varepsilon} T^{1+\varepsilon} p^{-n}.$$

We take

$$T = \begin{cases} x^{1/2} & \text{if } p^n \leq x^{1/3} \\ x^{1/4} p^{n/2} & \text{if } p^n > x^{1/3} \end{cases}.$$

Hence,

$$\begin{aligned} & \sum_{N(w) \leq x} \tau(w; \alpha, E_n^+) = \\ & = \operatorname{res}_{s=1} \left\{ \zeta_0(s; 0, 0) \left[\zeta_0\left(s; \frac{\alpha}{p^n}, 0\right) - \sum_{\mathbf{B}} N\left(\frac{\alpha}{p^n} + \beta\right)^{-s} \right] \left(\frac{x}{p^{2n}}\right)^s \frac{1}{s} \right\} + \\ & \quad + \sum_{\beta \in \mathbf{B}} \sum_{w \in \Omega(\alpha)} 1 + O\left(x^{1/2+3\varepsilon} p^{-n}\right). \end{aligned} \quad (8)$$

But we have

$$\sum_{\alpha \in E_n^+} \sum_{\beta \in \mathbf{B}} \sum_{w \in \Omega(\alpha)} 1 = \pi x \sum_{\alpha \in E_n^+} \sum_{\mathbf{B}} N(\alpha + \beta p^n)^{-1} + O\left(\left(\frac{x}{N(\alpha)}\right)^{1/3}\right) \quad (6)$$

$$\sum_{\alpha} \operatorname{res}_{s=1} \left\{ -\zeta_0(s) \sum_{\mathbf{B}} \frac{x^s}{s N(\alpha + \beta p^n)} \right\} = -\pi x \sum_{\alpha} \sum_{\mathbf{B}} \frac{1}{N(\alpha + \beta p^n)} \quad (7)$$

$$\begin{aligned} & \sum_{\alpha} \operatorname{res}_{s=1} \left\{ \zeta_0(s; 0, 0) \zeta_0\left(s; \frac{\alpha}{p^n}, 0\right) p^{-2ns} \frac{x^s}{s} \right\} = \\ & = \operatorname{res}_{s=1} \left[\left(\frac{\pi}{4} \frac{1}{s-1} + \left(\frac{\pi\gamma}{4} + L'(1, \chi_4) + \dots \right) \right) \times \right. \\ & \quad \left. \times \sum_{\alpha \in E_n^+} \frac{1}{|E_n^+|} \frac{1}{p^{2ns}} \left(\sum_{\hat{\chi} \in \hat{E}_n^+} \hat{\chi}(\alpha^{-1}) \zeta(s, \hat{\chi}) \right) \frac{x^s}{s} \right] \end{aligned} \quad (8)$$

where \hat{E}_n^+ is the group of characters for E_n^+ .

$$\zeta(s, \hat{\chi}) = \sum_{w \in G} \frac{\tilde{\chi}(w)}{N(w)^s}, \tilde{\chi} \in \hat{E}_n^+.$$

$$\zeta(s, \hat{\chi}) = \frac{\varepsilon(\hat{\chi})}{s-1} + b_0(\hat{\chi}) + b_1(\hat{\chi})(s-1) + \dots$$

$$\varepsilon(\tilde{\chi}) = \begin{cases} \frac{\pi}{4} \frac{p^2-1}{p^2} & \text{if } \tilde{\chi} = \tilde{\chi}_0 \text{ is the principal character from } \hat{E}_n^+; \\ 0 & \text{else.} \end{cases}$$

$$b_0(\tilde{\chi}_0) = \frac{\pi}{4} \frac{p^2-1}{p^2} \left(\gamma + \frac{L'(1, \chi_4)}{L(1, \chi_4)} + \frac{\log p^2}{p^2-1} \right) \quad (9)$$

Hence,

$$\begin{aligned} \operatorname{res}_{s=1} \left\{ \sum_{\alpha \in E_n^+} \zeta_0(s; 0, 0) \zeta_0\left(s; \frac{\alpha}{p^n}, 0\right) p^{-2ns} \frac{x^s}{s} \right\} = \\ = \frac{\pi^2}{2} \frac{p+1}{p} \frac{x \log x}{p^n} + \frac{\pi x}{4p^n} \frac{p+1}{p} (b_0(\tilde{\chi}_0) + \gamma + L'(1, \chi_4)) \end{aligned} \quad (10)$$

In the case $m \neq 0$ the proofs follows by analogous if we take $T = (xp^{-2n})^{\frac{1}{2}-2\varepsilon}$. ■

Well known lemma of Vinogradov on approximation of characteristic function of segment $\Delta \subset [0, 1]$ by truncated Fourier series gives the main result our paper.

Theorem 2. For $p^n \ll x^{\frac{1}{2}-\varepsilon}$ and $(\phi_2 - \phi_1) \ll (xp^{-2n})^{-\frac{1}{2}+\varepsilon}$ the following asymptotic formula

$$\sum_{\substack{\alpha \in E_n^+ \\ \varphi_1 \leq \arg w \leq \varphi_2 \\ N(w) \leq x}} \tau(w; \alpha, E_n^+) = \left(\frac{\pi^2 x \log x}{p^n} \cdot \frac{p+1}{p} + \frac{x}{p^{2n}} \sum_{\alpha \in E_n} c(\alpha, p^n) \right) (\varphi_2 - \varphi_1) + O(x^{\frac{1}{2}+\varepsilon}).$$

holds.

The constant in symbol "O" depends only on ε , $\varepsilon > 0$.

Concluding remark. The Theorem 2 establishes that the divisor function is an Gaussian integers with divisors from the norm group E_n^+ .

CONCLUSION. The divisor function in norm group E_n^+ was investigated. The asymptotic formula is obtained.

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Радова А. С.

Дільники Гаусових чисел на норменій підгрупі E_n^+

Резюме

Розглядається функція дільників на норменій підгрупі E_n^+ , де множина E_n^+ – мультиплікативна підгрупа мультиплікативної групи класів лишків модулю p^n над $\mathbb{Z}[i]$. Побудована асимптотична формула.

Ключові слова: гаусові числа, функція дільників, асимптотична формула.

Радова А. С.

Делители Гауссовых чисел на норменной подгруппе E_n^+

Резюме

Рассматривается функция делителей на норменной подгруппе E_n^+ , где множество E_n^+ – мультипликативная подгруппа мультипликативной группы классов вычетов по модулю p^n над $\mathbb{Z}[i]$. Построена асимптотическая формула.

Ключевые слова: гауссовы числа, функция делителей, асимптотическая формула.