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ON THE MULTIPLICATIVE PARTITION FUNCTION

We study the number of representations $n = s_1 \cdots s_m$, where s_j are sonor numbers, i.e. for every s_j there do not exist the natural numbers n and k such that $s_j = n^k$, $k \geqslant 2$. The counting function $f(n)$ of such representation is the multiplicative analogue of the additive partitions of n. We construct the asymptotic formula for summatory function of $f(n)$ and investigate the distribution of values of the generalized divisor function $L(n)$ (as the number of representations n factoring two sonor numbers).
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Introduction. Let $\mathcal{M}$ be a subset of integers with positive density, e. g.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n\in\mathcal{M},\\ n\leqslant N}} 1 > 0,$$

and $e(n)$ — its characteristic function. Multiplicative partition of an integer $n > 1$ is a representation of n as any product of numbers from $\mathcal{M}$, greater than 1. Number of such representations is denoted as $f^*(n, \mathcal{M})$ (or simply $f^*(n)$ if it is clear what $\mathcal{M}$ is selected). A sign $*$ shows that we mean the multiplicative partition. MacMahon [8] first studied a distribution of $f^*(n)$ at the set $\mathcal{M} = \mathbb{N}$, as multiplicative analog of Ramanujan partitions. He built an asymptotic formula for a sum $\sum_{n\leqslant x} f^*(n)$. Soon thereafter Oppenheim ([9], [10]) improved the result of MacMahon, obtaining a representation of the summation function $\sum_{n\leqslant x} f^*(n)$ as a series on values of a Bessel's function of the first kind $I_k(z)$:

$$\sum_{n\leqslant x} f^*(n) = x \sum_{k=0}^{\infty} d_k \frac{I_{k+1}(2\sqrt{\log x})}{\sqrt{\log x}^{k+1}} + O\left(x \frac{e^{\sqrt{\log x}}}{(\log x)^{\frac{3}{8}}} \right), \tag{1}$$

where d_k are the coefficients of Taylor's series expansion by the powers of a $(s-1)$ of the function $\frac{1}{s}F(s)e^{-\frac{1}{s-1}}$, where $F(s)$ – generating series of the sequence $\{f^*(n)\}$. Then other improvements of the remaining term in the formula (1) followed (see [5], [11], [7]), as well as various generalizations of a choice of the set $\mathcal{M}$(see [1], [12], [13], [4]). In [2], an order of the growth of $f^*(n)$ has been studied. The authors demonstrated that there is an infinite sequence of "highly factorable numbers" n, at which $f^*(n)$ takes maximal positive values:

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$$f^*(n) = n \cdot \exp\left(-\frac{\log n \log\log\log n}{\log\log n} + o(1)\right).$$

In a work of Warlimont[13], various examples of factorizations of integers are explored (e.g., different $\mathcal{M}$ sets).

In this paper, we will study another type of factorization of integers, which wasn't included in the list of the types of factorizations in [13].

Main Results

1. Statement of the problem. Let $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r} > 1$ and let $\alpha = GSD(a_1, a_2, \dots, a_r)$. We say that n is a "sonor" number (or integer non-power), if $\alpha = 1$. The unity (1) apparently is not classifiable as either sonor or integer power. We will denote the set of sonor numbers as S and integer powers as Q. Because $\mathbb{N} = S \cup Q$ and $S \cap Q = \emptyset$, taking into consideration that amount of integer powers $\leqslant x$ is $O(X^{1/2})$, the density of S is 1. Also, for convenience, we will use an expanded set of sonor numbers, S',

$$S' = S \cup \{1\}.$$

Denoting the number of sonors $\leqslant x$ as $k(x)$ and number of integer powers as $p(x)$, we can get

$$k(x) + p(x) + 1 = [x],$$

and therefore

$$k(x) = x - x^{\frac{1}{2}} - x^{\frac{1}{3}} + O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

A generating series $E(s)$ for sonor numbers

$$E(s) = \sum_{\substack{n=1,\\ n\in S}}^{\infty} \frac{1}{n^s} = \zeta(s) - \zeta(2s) - \zeta(3s) + g(s), \Re s > 1,$$

where $g(s)$ is regular in a half-plane $\Re s > \frac{1}{5}$ allows to investigate the function $k(x)$. Besides that, we can obtain an equality

$$\int_2^{\infty} \frac{k(x) x^{s-1}}{(x^s - 1)^2} dx = \frac{\zeta(s) - 1}{s} (\Re s > 1),$$

that is an interesting analogy of a well-known formula

$$\int_2^{\infty} \frac{\pi(x)}{x(x^s - 1)} dx = \frac{\ln \zeta(s)}{s} \ (\Re s > 1),$$

relating the function $\pi(x)$ and Reimann's zeta-function. In general, sonor numbers can be seen as an analogy for prime numbers (with prime numbers being a subset of sonors).

Further in this paper, we will explore some arithmetical functions associated with the sequence of sonor numbers.

Each integer number greater than 1 can be represented as a product of "expanded" sonor numbers (if 1 is included into the set of sonor numbers), but this representation is not unique. For example,

$$n = p_1^2 p_2^2 = p_1 \cdot p_1 \cdot p_2 \cdot p_2 = p_1(p_1 p_2^2) = p_2(p_2 p_1^2) = (p_1 \cdot p_2) \cdot (p_1 \cdot p_2).$$

Further we will take "expanded" S' as an example of $\mathcal{M}$ for the problems of multiplicative representations. Our attention will be focused on an investigation of three functions:

$$f^*(n) = \sum_{\substack{n=n_1\cdots n_k,\\ n_i\in S'}} 1, \quad f_0^*(n) = \sum_{\substack{n=n_1\cdots n_k,\\ n_i\in S',\\ 1<n_1<\cdots<n_k}} 1, \quad \hat{d}(n) = \sum_{\substack{n=n_1 n_2,\\ n_1,n_2\in S'}} 1.$$

2. Notation and supporting corollaries. Throughout we will use the following notation. The letter p denotes a prime number. We write $gcd(a,b) = (a,b)$ for the greatest common divisor of a and b. For any $t \in \mathbb{R}$ we write $\exp(t) = e^t$. For $s \in \mathbb{C}$ we denote $\Re s = \sigma$, $\Im s = t$, $s = \sigma + it$. $\zeta(s)$ is the Riemann zeta-function. $f(x) = O(g(x))$ means $|f(x)| \leqslant cg(x)$ for $x \geqslant x_0$ and some absolute constant $c > 0$. Here $f(x)$ is the complex function of the real x and $g(x)$ is a positive function of x for $x \geqslant x_0$. $f(x) \ll g(x)$ means the same as $f(x) = O(g(x))$. $f(x) = o(1)$ means that $\lim_{x\to\infty} f(x) = 0$.

Now we shall consider some assertions which will be necessary furthermore.

Corollary 7. *For $|t| \geqslant 3$ uniformly at σ we have*

$$\zeta(\sigma+it) \ll \begin{cases} 1 & if \quad \sigma \geqslant \frac{5}{4}, \\ \log|t| & if \quad 1 \leqslant \sigma \leqslant \frac{5}{4}, \\ |t|^{\frac{1-\sigma}{3}} \log|t| & if \quad \frac{1}{2} \leqslant \sigma < 1, \end{cases}$$

$$\zeta(1+it) \ll \log|t|.$$

Corollary 8. *For any $T \geqslant 3$ we have*

$$\int_1^T \left|\xi\left(\frac{1}{2}+it\right)\right|^2 dt \ll T\log T.$$

Those corollaries are well known.

Let then $e(n)$ be an arbitrary arithmetical function (not necessary a characteristic function of $\mathcal{M}$). We shall assume that $e(n) \geqslant 0$, $e(n) \leqslant n^\varepsilon$ for every small ε. Therefore, the series $\sum\limits_1^\infty \frac{e(n)}{n^s}$ absolutely converges in the half-plane $\Re s > 1$, and the following equality is true:

$$\prod_{n=2}^{\infty}\left(1-\frac{e(n)}{n^s}\right)^{-1} = \sum_{n=1}^{\infty}\frac{f^*(n)}{n^s}, \tag{2}$$

where $f^*(n) = \sum\limits_{n=n_1\cdots n_k} e(n_1\cdots e(n_k),\ f^*(1)=1.$

If $e(n)=0$ for $n\notin\mathcal{M}$, then

$$f^*(n) = \sum_{\substack{n=n_1\cdots n_k,\\ 1<n_i\in\mathcal{M}}} e(n_1)\cdots e(n_k),\ \ f^*(1)=1.$$

Let's denote as $f_0^*(n)$ the number of representation of n as a product of different elements $n_1,\ldots,n_k$, greater than 1, from $\mathcal{M}$, that is

$$f_0^*(n) = \sum_{\substack{n=n_1\cdots n_k,\\ 1<n_1<\cdots<n_k,\\ n_i\in\mathcal{M}}} e(n_1)\cdots e(n_k).$$

In this case we have

$$\prod_{n=2}^{\infty}\left(1+\frac{e(n)}{n^s}\right) = \sum_{n=1}^{\infty}\frac{f^*(n)}{n^s}. \tag{3}$$

If $\mathcal{M}$ is the expanded set of sonor numbers (including 1), functions $f^*(n)$ and $f_0^*(n)$, generally speaking, are not multiplicative. The function $\hat{d}(n)$ defined above as a number of representations of n as a product of two sonor numbers (including 1 as a potential co-factor), also is not multiplicative. Indeed, we have

$$\hat{d}\left(p^a\right) = \begin{cases} 1, & if \quad a=0\\ 2, & if \quad a=1\\ 1, & if \quad a=2,\\ 0, & if \quad a\geqslant 3.\end{cases}$$

However,

$$\hat{d}\left(p_1^2p_2^3\right) = \#\left\{\begin{matrix} p_1\cdot p_1p_2^3; & p_1p_2^3\cdot p_1; & p_1p_2^2\cdot p_1p_2;\\ p_1p_2\cdot p_1p_2^2; & p_1^2p_2^3\cdot 1; & 1\cdot p_1^2p_2^3\end{matrix}\right\} = 6,$$

$$\hat{d}\left(p_1^2\right)\cdot\hat{d}\left(p_2^3\right) = 0.$$

We will investigate the function $f_0^*(n)$ introduced above using a theorem proved by Y. Katai, M. Subbarao.

Corollary 9 ([7]**.** *, Th. 5.1] Let the sequences* $\{e(n)\}$ *and* $\{f(n)\}$ *satisfy to equation (3) with* $e(1)=f(1)=1$ *and let the function* $E(s)$ *given for* $\Re s>1$ *by the equation*

$$E(s) := \sum_{n=1}^{\infty}\frac{e(n)}{n^s}$$

satisfies for two assumptions

(i) there exist the positive constants A and β such that

$$E(s) = \frac{A}{(s-1)^{\beta}} + G(s),$$

where $G(s)$ be regular in the semi-plane $\Re s > \frac{1}{2}$;

(ii) there exists a positive constant A_0 such that

$$|E(1+it)| \leqslant A_0 \log|t|, \; if \; |t| \geqslant 3.$$

Then for any natural number N the asymptotic formula

$$T(x) := \sum_{n \leqslant x} f(n) = \exp\left(c_0 (\log x)^{\frac{\beta}{\beta+1}} \left\{ {\sum_{(h,\nu)}}^{*} H(h,\nu)(\log x)^{-\frac{2h+\nu\beta}{2\beta+2}} \times \right.\right.$$
$$\left.\left. \times \left[1 + c_0 (\log x)^{-\frac{1}{\beta+1}} - \frac{2h+\nu\beta}{2\beta}(\log x)^{-1} \right] + O\left((\log x)^{-\frac{2N+4+\beta}{2\beta+2}} \right) \right\} \right),$$

holds, where c_0 is a computable constant depended only on A and β; N is any fixed natural number; $H(h,\nu)$ are the suitable constants do not depended on x and N; the sign $*$ for $\sum\limits_{(h,\nu)}$ means that summation passes over all pairs (h,ν), $1 \leqslant h \leqslant N$, $\nu = 1, 2, \ldots$, for which $h + \frac{1}{2}\nu\beta \leqslant N + 2 + \frac{1}{2}\beta$.

3. Function of divisors $\hat{d}(n)$. The function of divisors $\hat{d}(n)$, as we mentioned above, is not multiplicative, dissimilar to the classical divisor function of Dirichlet. We have for $\Re s > 1$.

$$\sum_{n=1}^{\infty} \frac{\hat{d}(n)}{n^s} = \left(\sum_{m \in S'} \frac{1}{m^s} \right)^2 = \left(\sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{q \in Q} \frac{1}{q^s} \right)^2 := (\zeta(s) - g_0(s))^2 =$$
$$= \left(\zeta(s) - \sum_{n=1}^{\infty} \frac{1}{n^{2s}} + g_1(s) \right)^2, \tag{4}$$

where $g_1(s)$ is regular in the half-plane $\Re s > \frac{1}{3}$.

Theorem 1. *With $x \to \infty$, the following asymptotical formula is true:*

$$D(x) := \sum_{n \leqslant x} \hat{d}(n) = x \log x + A_1 x + O\left(x^{\frac{1}{2}} \log x \right)$$

with a computable constant A_1 and an absolute constant in the symbol "O".

Proof. The Perron's formula for the coefficient of Dirichlet series and the statement (4) yields:

$$D(x) =$$
$$= \frac{1}{2\pi i} \int\limits_{c-iT}^{c+iT} \left(\left(\zeta^2(s) - 2\zeta(s) \right) \left(\zeta(2s) - g_1(s) \right) + \left(\zeta(2s) - g_1(s) \right)^2 \right) \frac{x^s}{s} ds + \tag{5}$$
$$+ O\left(\frac{x^{1+\varepsilon}}{T} \right),$$

where $c > 1$, $T > 1$ will be chosen later.

Let's analyze a closed contour consisting of 8 parts:

$$\Gamma_0 : [c - iT,\ c + iT], \quad \Gamma_4 : \left[\tfrac{1}{4} - 3i,\ \tfrac{1}{4} + 3i\right],$$

$$\Gamma_1 : \left[\tfrac{1}{2} + iT,\ c + iT\right], \quad \Gamma_5 : \left[\tfrac{1}{4} - 3i,\ \tfrac{1}{2} - 3i\right],$$

$$\Gamma_2 : \left[\tfrac{1}{2} + 3i,\ \tfrac{1}{2} + iT\right], \quad \Gamma_6 : \left[\tfrac{1}{2} - iT,\ \tfrac{1}{2} - 3i\right],$$

$$\Gamma_3 : \left[\tfrac{1}{4} + 3i,\ \tfrac{1}{2} + 3i\right], \quad \Gamma_7 : \left[\tfrac{1}{2} - iT,\ c - iT\right].$$

In this case, the Cauchy's residue theorem yields:

$$\frac{1}{2\pi i}\int_{\Gamma_0} = + \frac{1}{2\pi i}\int_{\Gamma_1} + \frac{1}{2\pi i}\int_{\Gamma_2} + \frac{1}{2\pi i}\int_{\Gamma_3} + \frac{1}{2\pi i}\int_{\Gamma_4} - \\ - \frac{1}{2\pi i}\int_{\Gamma_5} - \frac{1}{2\pi i}\int_{\Gamma_6} - \frac{1}{2\pi i}\int_{\Gamma_7} + \operatorname*{res}_{s=\frac{1}{2}} + \operatorname*{res}_{s=1},$$

where integrated functions under all integrals, and also functions for which the residues are being determined, are equal to the function

$$\left(\left(\zeta^2(s) - 2\zeta(s)\right)\left(\zeta(2s) - g_1(s)\right) + \left(\zeta(2s) - g_1(s)\right)^2\right)\frac{x^s}{s}.$$

All the integrals, except integrals on contours Γ_3 and Γ_6, could be estimated using Corollary 1 as $O\left(x^2\right)$, and those integrals in its turn, following the Corollary 2, are estimated as

$$\left|\int_{\Gamma_3}\right| + \left|\int_{\Gamma_6}\right| \ll x^2 \log^2 x, \tag{6}$$

if we take $c = 1 + \frac{1}{\log x}$, $T = x^{\frac{3}{4}}$.

Besides that, it is easy to see that

$$\operatorname*{res}_{s=\frac{1}{2}} + \operatorname*{res}_{s=1} = x \log x + A_1 x, \tag{7}$$

where A_1 is a suitable constant.

From (5)-(4), the corollary's statement follows.

4. Functions $f^*(n)$ and $f_0^*(n)$ of multiplicative partitions of integers. To build an asymptotical formula for the average meaning of the function $f^*(n)$ introduced above, determining a number of multiplicative partitions of integers on the set of sonor numbers, we will use the Oppenheim's method [10].

Let's look at a modified Bessel's function of the first kind $I_n(z)$, $z \in \mathbb{C}$, $z \neq 0$, defined by a series

$$I_n(z) = \sum_{k=1}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k+n}}{\Gamma(k+1)\Gamma(k+n+1)}, \tag{8}$$

where $\Gamma(u)$ is the Euler's gamma function.

For a real positive x, the modified function $I_n(x)$ has an integral representation as

$$I_n(x) = \frac{x^n}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \frac{e^{s+\frac{x^2}{4s}}}{s^{n+1}} ds, \tag{9}$$

where c is any positive number.

Besides that, for $I_n(x)$ there is an asymptotic representation

$$I_n(x) = \frac{e^x}{\sqrt{2\pi x}} \left(1 - \frac{4n^2-1}{8x} + O\left(x^{-2}\right)\right). \tag{10}$$

(More about the function $I_n(x)$ see in [3]).

Theorem 2. *With $x \to \infty$, we have*

$$\sum_{n \ll x} f^*(n) = x \sum_{n=0}^{\infty} d_n \frac{I_{n+1}(2\sqrt{\log x})}{(\sqrt{\log x})^{n+1}} + O(x), \tag{11}$$

where coefficients d_n, $n = 0, 1, \ldots$ can be expressed through coefficients of Taylor's series on powers $(s-1)$ of a function, defined below in an equality (16).

Proof. Let $F(s)$ is the generating series for $f^*(n)$:

$$F(s) := \sum_{n=1}^{\infty} \frac{f^*(n)}{n^s}.$$

It is clear that for $\Re s > 1$ we have

$$F(s) = \prod_{n=2}^{\infty} \left(1 - \frac{e(n)}{n^s}\right)^{-1},$$

where $e(n)$ is a characteristic function of the set of expanded sonor numbers S.

From here,

$$\log F(s) = \sum_{m \in S'} \log\left(1 - \frac{1}{m^s}\right) = \sum_{m \in S'} m^{-s} + F_1(s), \tag{12}$$

where $F_1(s)$ is regular in a half-plane $\Re s > \frac{1}{2}$.

From (12) we conclude that $\log F(s) = \zeta(s) + F_2(s)$, where $F_2(s)$ is regular for $\Re s > \frac{1}{2}$. Therefore,

$$F(s) = \exp\left(\zeta(s) + F_2(s)\right) = \exp\left(\frac{1}{s-1} + F_3(s)\right),$$

where $F_3(s)$ is regular for $\Re s > \frac{1}{2}$ and in particular is a circle $|s-1| < \frac{1}{2}$.

Let we have in this circle a decomposition

$$\exp\left(F_3(s)\right) = \sum_{k=0}^{\infty} d_k (s-1)^k.$$

Then

$$F(s) = d_1 e^{\frac{1}{s-1}} \left(1 + b_1 (s-1) + b_2 (s-1)^2 + \cdots\right),$$

where $d_1 = e^{F_3(1)}$, $b_k = \frac{d_k}{d_1}$, $k = 2, 3, \ldots$.

For deriving an asymptotic formula for summation function $\sum\limits_{n \ll x} f^*(n)$, we will utilize Landau's method, building first an asymptotic representation of a sum

$$\sum_{n \ll x} f^*(n)(x-n),$$

and then based on asymptotic differentiation will find the necessary formula for the sum $\sum_{n \ll x} f^*(n)$.

We have

$$\sum_{n \ll x} f^*(n)(x-n) = \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \frac{F(s)}{s(s+1)} x^{s+1} ds,\ (c > 1).$$

Because $F(s)$ doesn't have singularities in the half-plane $\Re s \geqslant 1$ expect the point $s = 1$, we shall replace the integration contour $(c - i\infty,\ c + i\infty)$ by a composition of three contours,

$$\begin{aligned}
&\Gamma_1: \text{ line segment } (1 - i\infty,\ 1 - ia],\\
&\Gamma_2: \text{ semicircle of radius a } 1 + ae^{i\theta},\ -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2},\ 0 < a \leqslant 1,\\
&\Gamma_3: \text{ line segment } [1 + ia,\ 1 + i\infty).
\end{aligned}$$

From the estimation of $\zeta(s)$ on the unit line (see Corollary 1), we obtain that the integrals on Γ_1 and Γ_3 are evaluated as $O\left(x^2\right)$. On the semicircle Γ_2 we will make a change of variable $s = 1 + \frac{1}{z}$, so that $z = a^{-1} e^{i\theta}$ and θ changes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. We will contract the semicircle Γ_2 to a point. Then we get that for any $b > 0$

$$\begin{aligned}
&\frac{1}{2\pi i} \int_{\Gamma_2} F(s) \frac{x^{s+1}}{s(s+1)} ds = \frac{1}{2\pi i} \int_{\Gamma_2} F(s) \frac{x^{s-1+2}}{s(s+1)} ds =\\
&= c_1 x^2 \cdot \frac{1}{2\pi i} \int\limits_{b-i\infty}^{b+i\infty} \frac{x^{\frac{1}{z}} e^z}{(z+1)(2z+1)} e^{F_2\left(1+\frac{1}{z}\right)} dz + O\left(x^2\right) =\\
&= c_1 x^2 \cdot \frac{1}{2\pi i} \int\limits_{b-i\infty}^{b+i\infty} \frac{e^{z + \frac{\log x}{z}}}{(z+1)(2z+1)} \left(1 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots\right) dz + O\left(x^2\right).
\end{aligned}$$

Now, by virtue of a definition of the modified Bessel's function $I_n(z)$, we immediately get

$$\frac{1}{2\pi i}\int_{\Gamma_2} F(s)\frac{x^{s+1}}{s(s+1)}ds =$$
$$= \frac{c_1 x^2}{2}\cdot\frac{1}{2\pi i}\int\limits_{b-i\infty}^{b+i\infty}\frac{e^{z+\frac{\log x}{z}}}{z^2}\left(1+\frac{b_1'}{z}+\frac{b_2'}{z^2}+\cdots\right)dz + O\left(x^2\right) =$$
$$= \frac{c_1 x^2}{2}\sum_{n=0}^{\infty} b_n' \int\limits_{b-i\infty}^{b+i\infty} e^{z+\frac{\log x}{z}} z^{-n-2}dz + O\left(x^2\right) =$$
$$= \frac{c_1 x^2}{2}\sum_{n=0}^{\infty} b_n I_{n+1}(2\sqrt{\log x})(\log x)^{-\frac{n+1}{2}} + O\left(x^2\right).$$

Now, using asymptotic differentiation, we come to the statement of the theorem.

Note. The relation (10) shows that the asymptotic formula, obtained in the Theorem 2, is nontrivial.

Now we are going over to an investigation of the sum:

$$D_0(x) := \sum_{n\leqslant x} f_0^*(n).$$

From (3) it follows that for $\Re s > 1$

$$E(s) = \sum_{n=1}^{\infty}\frac{e(n)}{n^s} = \sum_{m\in S'}\frac{1}{m^s} = \zeta(s) + F_0(s),$$

where $F_0(s)$ is regular in a half-plane $\Re s > \frac{1}{2}$.

Therefore, all conditions of the Katai-Subbarao theorem are satisfied with a parameter $\beta = 1$. Hence the following assertion is true.

Theorem 3. *Let $f_0^*(n)$ be a number of representation of n as a product of sonor numbers. Then,*

$$D_0(x) = e^{c_0\sqrt{\log x}}\left\{\sum\nolimits_{(h,v)}^{*} H(h,v)(\log x)^{-\frac{2h+v}{4}}\left(1 + c_0(\log x)^{-\frac{1}{2}} - \tfrac{2h+v}{2}(\log x)^{-1}\right) + \right.$$
$$\left. + O\left((\log x)^{-\frac{2N+5}{4}}\right)\right\},$$

where sign $*$ at the sum $\sum\limits_{(h,v)}$ means that the summation is performed for all pairs (h,v), $1\leqslant h\leqslant N$, $v = 1, 2\ldots$, for which $h + \frac{1}{2}v \leqslant N + \frac{5}{2}$.

Conclusion. The proved theorems show that the problems of a multiplicative partition of integer numbers on the set of sonor numbers can be researched by methods of investigation of similar problems for multiplicative partitions on the set of $\mathbb{N}$, and it is possible to assume the correctness of an analogy regarding the maximum order of the functions $f^*(n)$ and $f_0^*(n)$.

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Про мультиплікативну функцію розбиття

Резюме

Ми вивчаємо кількість зображень $n = s_1 \cdots s_m$, де s_j — сонорні числа, тобто для кожного s_j не існує натуральних чисел n і k, таких що $s_j = n^k$, $k \geqslant 2$. Зчитуюча функція $f(n)$ таких зображень є мультиплікативним аналогом адитивної функції розбиттів n. Ми будуємо асимптотичну формулу для суматорної функції для $f(n)$ і досліджуємо розподілення значень узагальненої функції дільників $L(n)$ (кількість зображень n у вигляді добутку двох сонорних чисел).

Ключові слова: мультиплікативна функція розбиттів, сонорні числа, асимптотична формула, ряди Діріхле .

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О мультипликативной функции разбиения

Резюме

Мы изучаем количество представлений $n = s_1 \cdots s_m$, где s_j — сонорные числа, т. е. для каждого s_j не существуют натуральные числа n и k, такие что $s_j = n^k$, $k \geqslant 2$. Считывающая функция $f(n)$ таких представлений является мультипликативным аналогом аддитивной функции разбиения n. Мы строим асимптотическую формулу для сумматорной функции для $f(n)$ и исследуем распределение значений обобщенной функции делителей $L(n)$ (количество представлений n в виде произведения двух сонорных чисел).

Ключевые слова: мультипликативная функция разбиений, сонорные числа, асимптотическая формула, ряды Дирихле .