UDC 517.925.52, 517.929

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OSCILLATION CRITERIA FOR HIGHER ORDER SUBLINEAR DELAY DIFFERENTIAL EQUATIONS

The results of this paper were reported at International Workshop "Nonlinear Analysis and Nonautonomous Ordinary Differential Equations", Odesa, Ukraine, June 23-27, 2017.

In the interval $[a,+\infty[$, the sublinear differential equations of order $n \geqslant 2$ are considered. A solution of such equation is called proper if it is not identically equal to zero in any neighbourhood of $+\infty$. The proper solution is called oscillatory if it changes its sign in any neighbourhood of $+\infty$. We say that the equation has property A if every its proper solution for n even is oscillatory, and for n odd either oscillatory or monotone and vanishing at infinity together with their derivatives up to order $n-1$, inclusive.

To investigate oscillatory properties of the above-mentioned equations the sets $M_{sub}([a,+\infty[\times\mathbb{R})$ and $\widetilde{M}_{sub}([a,+\infty[\times\mathbb{R})$ of sublinear with respect to the second argument continuous functions $f : [a,+\infty[\times\mathbb{R} \to \mathbb{R}$ are introduced (see Definitions 1 and 2).

In the case when $f \in M_{sub}([a,+\infty[\times\mathbb{R})$, for the differential equation

$$u^{(n)}(t) = f\big(t, u(\tau(t))\big)$$

the criterion of the existence of property A and in the case when $f \in \widetilde{M}_{sub}([a,+\infty[\times\mathbb{R})$ the criterion of oscillation of all proper solutions are established.

As an example, the delay differential equation

$$u''(t) = p(t)\big[\ln(1 + |u(\tau(t))|)\big]^{\mu} \operatorname{sgn}(u(\tau(t)))$$

is considered, where μ is a positive constant, while $p : [a,+\infty[\to]-\infty,0]$ and $\tau : [a,+\infty[\to [1,+\infty[$ are continuous functions. It is stated that if n is even, or n is odd and

$$\limsup_{t\to+\infty} \frac{\tau(t)}{t} < 1,$$

then for all proper solutions of that equation to be oscillatory, it is necessary and sufficient that the equality

$$\int\limits_a^{+\infty} [\ln(\tau(t))]^{\mu} p(t)\, dt = -\infty$$

be satisfied.

MSC: 34C10, 34K11.
Key words: delay differential equation, nonautonomous, sublinear, proper solution, oscillatory solution, property A, oscillation criterion.
DOI: 10.18524/2519–206x.2018.1.134625.

Introduction. The problem on the oscillation of solutions of nonautonomous ordinary differential and functional differential equations has long been attracting the

Received 03.12.2017 © Kiguradze I., Kiguradze T., 2018

attention of mathematicians and is the subject of numerous studies (see, e.g., [1–18] and the references therein). Nevertheless, this problem for some classes of sublinear equations still remains unsolved. The results of the present paper fill up to some extent the existing gap.

On the infinite interval $[a,+\infty[$ consider the differential equation

$$u^{(n)}(t) = f(t, u(\tau(t))), \tag{1}$$

where $n \geqslant 2$, $a > 1$, while $f : [a,+\infty[\times\mathbb{R} \to \mathbb{R}$ and $\tau : [a,+\infty[\to \mathbb{R}$ are continuous functions such that

$$f(t,x)x \leqslant 0 \ \text{ for } \ t \geqslant a, \ \ x \in \mathbb{R}, \tag{2}$$

$$1 < \tau(t) \leqslant t \ \text{ for } \ t \geqslant a, \quad \lim_{t\to+\infty} \tau(t) = +\infty. \tag{3}$$

Let $t_0 \geqslant a$. The n-times continuously differentiable function $u : [t_0,+\infty[\to \mathbb{R}$ is said to be a **solution of equation** (1) **defined on the interval** $[t_0,+\infty[$ if there exists a continuous function $u_0 :]-\infty, t_0]$ such that

$$u^{(n)}(t) = f\big(t, w(u)(\tau(t))\big) \ \text{ for } \ t \geqslant t_0,$$

where

$$w(u)(t) = \begin{cases} u(t) & \text{for } \ t > t_0, \\ u_0(t) & \text{for } \ t \leqslant t_0. \end{cases}$$

A solution u of equation (1) defined on some infinite interval $[t_0,+\infty[\subset [a,+\infty[$ is said to be **proper** if it is not identically equal to zero in any neighborhood of $+\infty$.

A proper solution $u : [t_0,+\infty[\to \mathbb{R}$ of equation (1) is said to be:

- **oscillatory** if it changes its sign in any neighbourhood of $+\infty$ and **nonoscillatory**, otherwise;
- **Kneser solution** if there exists $t_1 \geqslant t_0$ such that

$$(-1)^{i-1} u^{(i-1)}(t) u(t) \geqslant 0 \ \text{ for } \ t \geqslant t_1;$$

- **vanishing at infinity** if

$$\lim_{t\to+\infty} u(t) = 0.$$

Following [16], we say that equation (1) has **property** A if every its proper solution for n even is oscillatory, and for n odd either is oscillatory or is vanishing at infinity Kneser solution.

Besides these definitions generally accepted in the oscillation theory, we apply also the following two definitions.

Definition 1. *The function* $f : [a,+\infty[\times\mathbb{R} \to \mathbb{R}$ *belongs to the set* $M_{sub}([a,+\infty[\times\mathbb{R})$ *if there exist numbers* $\lambda \in [0,1[$, $r_0 > 0$, $r \geqslant 1$ *and a nondecreasing function* $\delta :]0, r_0] \to]0, 1]$ *such that*

$$|f(t,y)| \geqslant \delta(|x|)|f(t,x)| \ \text{ for } \ t \geqslant a, \ \ xy > 0, \ \ |x| \leqslant |y| \leqslant r_0,$$

$$|y|^{-\lambda}|f(t,y)| \leqslant r|x|^{-\lambda}|f(t,x)| \ \text{ for } \ t \geqslant a, \ \ xy > 0, \ \ |y| \geqslant |x| \geqslant r_0.$$

Definition 2. *The function* $f : [a,+\infty[\times\mathbb{R} \to \mathbb{R}$ *belongs to the set* $\widetilde{M}_{sub}([a,+\infty[\times\mathbb{R})$ *if there exist numbers* $\lambda\in]0,1[$ *and* $r\geqslant 1$ *such that*

$$|y|^{-\lambda}|f(t,y)|\leqslant r|x|^{-\lambda}|f(t,x)| \ \text{ for } \ t\geqslant a, \ \ xy>0, \ \ |y|\geqslant|x|.$$

The oscillatory criteria below deal with the cases, where

$$f\in M_{sub}([a,+\infty[\times\mathbb{R}), \tag{4}$$

or

$$f\in \widetilde{M}_{sub}([a,+\infty[\times\mathbb{R}). \tag{5}$$

In both cases we have

$$\lim_{|x|\to+\infty}\frac{f(t,x)}{x}=0.$$

Consequently, equation (1) is sublinear.

The above-considered classes of sublinear equations contain, for example, the equations

$$u^{(n)}(t)=\sum_{k=1}^{m}p_k(t)\big(1+|u(\tau(t))|\big)^{\mu_k}|u(\tau(t))|^{\lambda_k}\operatorname{sgn}(u(\tau(t))), \tag{6}$$

$$u^{(n)}(t)=p(t)\big[\ln\big(1+|u(\tau(t))|\big)\big]^{\mu}\operatorname{sgn}(u(\tau(t))), \tag{7}$$

$$u^{(n)}(t)=p(t)\exp\big(-|u(\tau(t))|\big)|u(\tau(t))|^{\mu}\operatorname{sgn}(u(\tau(t))), \tag{8}$$

where $p_k:[a,+\infty[\to]-\infty,0]$ $(k=1,\dots,m)$, $p:[a,+\infty[\to]-\infty,0]$ and $\tau:[a,+\infty[\to \mathbb{R}$ are continuous functions, while λ_k, μ_k $(k=1,\dots,m)$ and μ are constants.

Theorem 1. *Let conditions* (2)–(4) *be fulfilled. Then equation* (1) *has property* A *if and only if the equalities*

$$\int_a^{+\infty}t^{n-1}|f(t,x)|\,dt=+\infty, \quad \int_a^{+\infty}\big|f\big(t,[\tau(t)]^{n-1}x\big)\big|\,dt=+\infty \ \text{ for } \ x\neq 0 \tag{9}$$

hold.

Remark 1. According to condition (4) the equality

$$\int_a^{+\infty}|f(t,[\tau(t)]^{n-1}x)|\,dt=+\infty \ \text{ for } \ x\neq 0 \tag{10}$$

implies the equality

$$\int_a^{+\infty}t^{n-1}|f(t,x)|\,dt=+\infty \ \text{ for } \ |x|\geqslant r_0$$

for some sufficiently large $r_0 > 0$. However, the equality

$$\int_a^{+\infty} t^{n-1}|f(t,x)|\,dt = +\infty \quad \text{for} \quad x \neq 0$$

does not follow from (10). Consequently, in Theorem 1 condition (9) cannot be replaced by condition (10).

As an example, consider the case, where

$$f(t,x) = \begin{cases} g(t)x & \text{for } |x| \leqslant 1, \\ g(t)\big[1+(|x|-1)|g(t)|\exp(t)\big]^{-1} + h(t)(|x|-1)^{\lambda}\operatorname{sgn} x & \text{for } 1<|x|<2, \\ g(t)\big[1+(|x|-1)|g(t)|\exp(t)\big]^{-1} + h(t)\Big(\dfrac{|x|}{2}\Big)^{\lambda}\operatorname{sgn} x & \text{for } |x| \geqslant 2, \end{cases} \tag{11}$$

where $\lambda \in \,]0,1[\,$, and $g : [a,+\infty[\, \to \,]-\infty,0]$ and $h : [a,+\infty[\, \to \,]-\infty,0]$ are continuous functions. Then condition (9) is equivalent to the condition

$$\int_a^{+\infty} t^{n-1}g(t)\,dt = -\infty, \qquad \int_a^{+\infty} [\tau(t)]^{(n-1)\lambda}h(t)\,dt = -\infty, \tag{12}$$

whereas condition (10) is equivalent to the condition

$$\int_a^{+\infty} [\tau(t)]^{(n-1)\lambda}h(t)\,dt = -\infty.$$

Remark 2. If the function f is of form (11), then, by Theorem 1, equation (1) has property A if and only if equalities (12) hold.

From the above theorem, for equations (6)–(8) we have the following corollaries.

Corollary 1. *Let*

$$\lambda_k > 0, \quad \mu_k + \lambda_k < 1 \quad (k = 1,\dots,m)$$

and condition (3) *hold. Then equation* (6) *has property A if and only if*

$$\int_a^{+\infty} \bigg(\sum_{k=1}^{n} \int_a^{+\infty} [\tau(t)]^{(n-1)}(\lambda_k + \mu_k)p_k(t)\bigg)\,dt = -\infty. \tag{13}$$

Corollary 2. *Let $\mu > 0$ and condition* (3) *hold. Then equation* (7) *has property A if and only if*

$$\int_a^{+\infty} [\ln(\tau(t))]^{\mu}p(t)\,dt = -\infty. \tag{14}$$

Corollary 3. *Let $\mu > 0$ and condition* (3) *hold. Then equation* (8) *has property A if and only if*

$$\int_a^{+\infty} [\tau(t)]^{(n-1)\mu} \exp\big(-x[\tau(t)]^{n-1}\big)p(t)\,dt = -\infty \ \ \textit{for} \ \ x \neq 0. \tag{15}$$

Remark 3. If

$$\liminf_{t\to+\infty} \frac{[\tau(t)]^{n-1}}{\ln t} > 0$$

and the function

$$p_0(t) = [\tau(t)]^{1-n} \ln\big([\tau(t)]^{(n-1)\mu}|p(t)|\big)$$

is nondecreasing, then (15) is fulfilled if and only if

$$\lim_{t\to+\infty} p_0(t) = +\infty.$$

The following theorem is specific for differential equations with delay and concerns the case where the function τ satisfies the condition

$$1 < \tau(t) < t \ \ \text{for} \ \ t > a, \quad \lim_{t\to+\infty} \tau(t) = +\infty, \quad \limsup_{t\to+\infty} \frac{\tau(t)}{t} < 1. \tag{16}$$

Theorem 2. *Let n be odd and conditions* (2), (5) *and* (16) *be fulfilled. Then every proper solution of equation* (1) *is oscillatory if and only if equality* (10) *holds.*

Unlike the oscillation theorems of Chanturia–Koplatadze [17], Theorems 1 and 2 cover the case, where

$$\lim_{|x|\to+\infty} \frac{f(t,x)}{|x|^\varepsilon} = 0 \ \ \text{for any} \ \ \varepsilon > 0,$$

or

$$\lim_{|x|\to+\infty} |x|^\gamma f(t,x) = 0 \ \ \text{for any} \ \ \gamma > 0.$$

From Theorem 2, just as from Theorem 1, follow new oscillation criteria for equations (6)–(8).

Corollary 4. *Let n be odd,*

$$0 < \lambda_k < 1, \ \ \mu_k + \lambda_k < 1 \ \ (k = 1, \dots, m),$$

and condition (16) *be fulfilled. Then every proper solution of equation* (6) *is oscillatory if and only if equality* (13) *holds.*

Corollary 5. *Let n be odd,*

$$0 < \mu < 1, \tag{17}$$

and condition (16) *be fulfilled. Then every proper solution of equation* (7) *is oscillatory if and only if equality* (14) *holds.*

Corollary 6. *Let n be odd and conditions* (16), (17) *be fulfilled. Then every proper solution of equation* (8) *is oscillatory if and only if equality* (15) *holds.*

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Кігурадзе І., Кігурадзе Т.
Критерій осциляції для сублінійних диференціальних рівнянь із запізненням

Резюме

Розглядається сублінійне диференціальне рівняння порядку $n \geqslant 2$ на інтервалі $[a, +\infty[$. Розв'язок такого рівняння називається правильним, якщо він не дорівнює нулеві в будь-якому околі $+\infty$. Правильний розв'язок називається осцилюючим, якщо він змінює свій знак в будь-якому околі $+\infty$. Казатимемо, що рівняння володіє властивістю A, якщо кожний його правильний розв'язок для парного n є осцилюючим, а для непарного n або осцилюючим, або таким, що монотонно наближається до нуля на нескінченності разом із своїми похідними до $n-1$-го порядку включно.

Щоб дослідити осциляційні властивості таких рівнянь, вводяться дві множини — $M_{sub}([a, +\infty[\times\mathbb{R})$ та $\widetilde{M}_{sub}([a, +\infty[\times\mathbb{R})$ — сублінійних за другим аргументом неперервних функцій $f : [a, +\infty[\times\mathbb{R} \to \mathbb{R}$.

У випадку, коли $f \in M_{sub}([a, +\infty[\times\mathbb{R})$, для диференціального рівняння

$$u^{(n)}(t) = f\big(t, u(\tau(t))\big)$$

отримано критерій існування властивості A, а для випадку, коли $f \in \widetilde{M}_{sub}([a, +\infty[\times\mathbb{R})$, отримано критерій осциляції усіх правильних розв'язків.

В якості прикладу розглянуто диференціальне рівняння із запізненням

$$u''(t) = p(t)\big[\ln(1 + |u(\tau(t))|)\big]^{\mu} \operatorname{sgn}(u(\tau(t))),$$

де μ є позитивною константою, $p : [a, +\infty[\to]-\infty, 0]$ та $\tau : [a, +\infty[\to [1, +\infty[$ є неперервними функціями. Стверджується, що, якщо n парне або n непарне та одночасно

$$\limsup_{t\to+\infty} \frac{\tau(t)}{t} < 1,$$

то для того, щоб усі правильні розв'язки рівняння були осцилюючими, необхідно і достатньо виконання рівності

$$\int\limits_a^{+\infty} [\ln(\tau(t))]^{\mu} p(t)\, dt = -\infty.$$

Ключові слова: диференціальне рівняння із запізненням, неавтономне, сублінійне, правильний розв'язок, осцилюючий розв'язок, властивість A, критерій осциляції .

Кигурадзе И., Кигурадзе Т.
Критерий осцилляции для сублинейных дифференциальных уравненний с запаздыванием

Резюме

Рассматривается сублинейное дифференциальное уравнение порядка $n \geqslant 2$ на интервале $[a, +\infty[$. Решение такого уравнения называется правильным, если оно не равно нулю в любой окрестности $+\infty$. Правильное решение называется осциллирующим, если оно меняет свой знак в любой окрестности $+\infty$. Будем говорить, уравнение обладает свойством A, если каждое его правильное решение для четного n является осциллирующим, а для нечетного n либо осциллирующим, либо стремящимся к нулю на бесконечности вместе со своими производными до $n-1$-го порядка включительно.

Чтобы исследовать осцилляционные свойства таких уравнений, вводятся два множества — $M_{sub}([a,+\infty[\times\mathbb{R})$ и $\widetilde{M}_{sub}([a,+\infty[\times\mathbb{R})$ — сублинейных по второму аргументу непрерывных функций $f:[a,+\infty[\times\mathbb{R}\to\mathbb{R}$.

В случае, когда $f\in M_{sub}([a,+\infty[\times\mathbb{R})$, для дифференциального уравнения

$$u^{(n)}(t)=f\big(t,u(\tau(t))\big)$$

получен критерий выполнения условия A, а для случая, когда $f\in\widetilde{M}_{sub}([a,+\infty[\times\mathbb{R})$, получен критерий осцилляции всех правильных решений.

В качестве примера рассмотрено дифференциальное уравнение с запаздыванием

$$u''(t)=p(t)\big[\ln(1+|u(\tau(t))|)\big]^{\mu}\operatorname{sgn}(u(\tau(t))),$$

где μ — положительная константа, $p:[a,+\infty[\to]-\infty,0]$ и $\tau:[a,+\infty[\to[1,+\infty[$ являются непрерывными функциями. Утверждается, что, если либо n четное, либо n нечетное и одновременно

$$\limsup_{t\to+\infty}\frac{\tau(t)}{t}<1,$$

то для того, чтобы все правильные решения уравнения были осциллирующими, необходимо и достаточно выполнения равенства

$$\int_a^{+\infty}[\ln(\tau(t))]^{\mu}p(t)\,dt=-\infty.$$

Ключевые слова: дифференциальное уравнение с запаздыванием, неавтономное, сублинейное, правильное решение, осциллирующее решение, свойство A, критерий осцилляции .